Integral Calculus

3.9 Differentials

The tangent line to the graph of f at the point $(c, f(c))$, according to the point-slope equation, is
$y - f(c) = f'(c)(x - c)$ or, rearranged,
$y = f(c) + f'(c)(x - c)$, where we call the change in x, $x - c = \Delta x$.

Actual change in y is given by $\Delta y = f(c + \Delta x) - f(c)$.
When Δx is small, change in y can be approximated by $\Delta y \approx f'(c)\Delta x$.

For such an approximation, Δx is denoted dx, and is called the differential of x.

For a differentiable function $y = f(x)$, the differential of y is $\boxed{dy = f'(x)dx}$

The approximate function value at $c + \Delta x$ can be found by $\boxed{f(c + \Delta x) \approx f(c) + f'(c)\Delta x}$

Differential Formulas: For differentiable functions of x, u and v,

1. $d[cu] = cdu$
2. $d[u \pm v] = du \pm dv$
3. $d[uv] = udv + vdu$
4. $d\left[\frac{u}{v}\right] = \frac{vdu - udv}{v^2}$

4.1 Antiderivatives

$F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$ for all x on a given interval.

A synonym for antiderivative is indefinite integral, and if $F'(x) = f(x)$, we write $\int f(x)dx = F(x) + C$.

Power rule: $\int x^n dx = \frac{x^{n+1}}{n+1} + C\,,\ n \neq -1$

For vertical motion problems,

acceleration due to gravity $a(t) = -32\ ft/s^2$ or $a(t) = -9.8\ m/s^2$
the integral of acceleration is velocity $(t) = at + v_o$, where v_o is the initial velocity $v(0)$
the integral of velocity is position $s(t) = \frac{1}{2}at^2 + v_o t + s_o$, where s_o is the initial position $s(0)$

$s(t) = position\ at\ time\ t$
$v(t) = s'(t) = velocity\ at\ time\ t$
$a(t) = v'(t) = s''(t) = acceleration\ at\ time\ t$

Integration formulas:

$$\int 0\,dx = C$$

$$\int k\,dx = kx + C$$

$$\int kf(x)dx = k\int f(x)dx$$

$$\int [f(x) \pm g(x)]dx = \int f(x)\,dx \pm \int g(x)\,dx$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C\,, n \neq -1$$

$$\int \cos x\ dx = \sin x + C$$

$$\int \sin x\ dx = -\cos x + C$$

$$\int \sec^2 x\ dx = \tan x + C$$

$$\int \csc^2 x\ dx = -\cot x + C$$

$$\int \sec x \tan x\ dx = \sec x + C$$

$$\int \csc x \cot x\ dx = -\csc x + C$$

4.2 Area

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

$$\sum_{i=1}^{n} c = cn$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + \frac{b-a}{n} i\right) \cdot \frac{b-a}{n}$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

4.4 – The Fundamental Theorem of Calculus

1st Fundamental Theorem of Calculus:
If a function f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Mean Value Theorem for Integrals:
If a function f is continuous on the closed interval $[a, b]$, then there exists a number c in the closed interval $[a, b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b - a)$$

Recall the Mean Value Theorem:
If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Average value of a function on an interval:
If f is integrable on the closed interval $[a, b]$, then the average value of f on the interval is

$$\frac{1}{b - a}\int_a^b f(x)\,dx$$

The Second Fundamental Theorem of Calculus:
If f is continuous on an open interval I containing a, then, for every x in the interval,

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

$$\frac{d}{dx}\left[\int_a^{g(x)} f(t)\,dt\right] = f(g(x)) \cdot g'(x)$$

Examples:

Find the value of c guaranteed by the Mean Value Theorem for Integrals for the function over the indicated interval.

48. $f(x) = \cos x$; $[-\pi/3, \pi/3]$

Find the average value of the function over the interval and all values of x in the interval for which the function equals its average value.

52. $f(x) = \cos x$; $[0, \pi/2]$

Find $F'(x)$.

82. $F(x) = \int_1^x \frac{t^2}{t^2+1} dt$

$$F(x) = \int_{x^2}^{3} \cos t \, dt$$

<u>**4.5 – Integration by Substitution**</u>

Find the indefinite integral.

20. $\int \frac{x^3}{(1+x^4)^2} dx$

46. $\int x \sin x^2 \, dx$

60. $\int (x+1)\sqrt{2-x} dx$

Evaluate the definite integral.

72. $\int_0^2 x \sqrt[3]{4+x^2} dx$

92. $\int_{-\pi/2}^{\pi/2} \sin^2 x \cos x \, dx$

5.2, 5.4, 5.5 – Logarithmic and Exponential Integration

$$\int \frac{du}{u} = \ln|u| + c$$

$$\int \frac{f'(x)dx}{f(x)} = \ln|f(x)| + c$$

$$\int \tan u \, du = -\ln|\cos u| + c$$

$$\int \cot u \, du = \ln|\sin u| + c$$

$$\int \sec u \, du = \ln|\sec u + \tan u| + c$$

$$\int \csc u \, du = -\ln|\csc u + \cot u| + c$$

$$\int e^x dx = e^x + c$$

$$\int a^x dx = \frac{a^x}{\ln a} + c$$

Examples:

Find the indefinite integral.

18. $\int \frac{x^3-3x^2+4x-9}{x^2+3} dx$

20. $\int \frac{1}{x\ln(x^3)} dx$

Solve the differential equation.

38. $\frac{dy}{dx} = \frac{2x}{x^2-9}$, $(0,4)$

Evaluate the definite integral.

46.

48.

Find $F'(x)$.

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Find the average value of the function over the interval.

80. $f(x) = \sec\frac{\pi x}{6}$, $[0,2]$

5.9 Inverse Trig Functions

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\frac{u}{a} + c$$

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\arctan\frac{u}{a} + c$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\operatorname{arcsec}\frac{|u|}{a} + c$$

6.2-6.4 – Volume, Arc Length and Surface Area

Arc length $= \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$

Volume $= \int_a^b \pi r^2 dx$

Surface area $= \int_a^b 2\pi r\sqrt{1 + [f'(x)]^2}\, dx$

7.1 – Basic Integration Rules

(chart on page 485)

7.2 – Integration by Parts

7.3 – Trigonometric Integrals

7.4 – Trigonometric Substitution

7.5 – Partial Fractions

6.5 - Work

Application: Work as integral of Force

5.7 – Solving Differential Equations by Separation of Variables

Find a particular solution to the differential equation based on the given initial conditions.