Integral Calculus

3.9 Differentials

The tangent line to the graph of f at the point $(c, f(c))$, according to the point-slope equation, is
$y - f(c) = f'(c)(x - c)$ or, rearranged,
$y = f(c) + f'(c)(x - c)$, where we call the change in x, $x - c = \Delta x$.

Actual change in y is given by $\Delta y = f(c + \Delta x) - f(c)$.
When Δx is small, change in y can be approximated by $\Delta y \approx f'(c)\Delta x$.

For such an approximation, Δx is denoted dx, and is called the <u>differential of x</u>.

For a differentiable function $y = f(x)$, the <u>differential of y</u> is $\boxed{dy = f'(x)dx}$

The <u>approximate function value</u> at $c + \Delta x$ can be found by $\boxed{f(c + \Delta x) \approx f(c) + f'(c)\Delta x}$

<u>Differential Formulas</u>: For differentiable functions of x, u and v,

1. $d[cu] = cdu$
2. $d[u \pm v] = du \pm dv$
3. $d[uv] = udv + vdu$
4. $d\left[\frac{u}{v}\right] = \frac{vdu - udv}{v^2}$

4.1 Antiderivatives

$F(x)$ is an <u>antiderivative</u> of $f(x)$ if $F'(x) = f(x)$ for all x on a given interval.

A synonym for antiderivative is <u>indefinite integral</u>, and if $F'(x) = f(x)$, we write $\int f(x)dx = F(x) + C$.

<u>Power rule</u>: $\int x^n dx = \frac{x^{n+1}}{n+1} + C\,,\ n \neq -1$

For vertical motion problems,

<u>acceleration</u> due to gravity $a(t) = -32\ ft/s^2$ or $a(t) = -9.8\ m/s^2$
the integral of acceleration is <u>velocity</u> $(t) = at + v_o$, where v_o is the <u>initial velocity</u> $v(0)$
the integral of velocity is <u>position</u> $s(t) = \frac{1}{2}at^2 + v_o t + s_o$, where s_o is the <u>initial position</u> $s(0)$

$s(t) = position\ at\ time\ t$
$v(t) = s'(t) = velocity\ at\ time\ t$
$a(t) = v'(t) = s''(t) = acceleration\ at\ time\ t$

Integration formulas:

$$\int 0\, dx = C$$

$$\int k\, dx = kx + C$$

$$\int kf(x)dx = k\int f(x)dx$$

$$\int [f(x) \pm g(x)]dx = \int f(x)\, dx \pm \int g(x)\, dx$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C\, , n \neq -1$$

$$\int \cos x\ dx = \sin x + C$$

$$\int \sin x\ dx = -\cos x + C$$

$$\int \sec^2 x\ dx = \tan x + C$$

$$\int \csc^2 x\ dx = -\cot x + C$$

$$\int \sec x \tan x\ dx = \sec x + C$$

$$\int \csc x \cot x\ dx = -\csc x + C$$

4.2 Area

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

$$\sum_{i=1}^{n} c = cn$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + \frac{b-a}{n} i\right) \cdot \frac{b-a}{n}$$

$$\int_a^a f(x)dx = 0$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

4.4 – The Fundamental Theorem of Calculus

1st Fundamental Theorem of Calculus:
If a function f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Mean Value Theorem for Integrals:
If a function f is continuous on the closed interval $[a, b]$, then there exists a number c in the closed interval $[a, b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

Recall the Mean Value Theorem:
If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Average value of a function on an interval:
If f is integrable on the closed interval $[a, b]$, then the average value of f on the interval is

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

The Second Fundamental Theorem of Calculus:
If f is continuous on an open interval I containing a, then, for every x in the interval,

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

$$\frac{d}{dx}\left[\int_a^{g(x)} f(t)\,dt\right] = f(g(x)) \cdot g'(x)$$

5.2, 5.4, 5.5 – Logarithmic and Exponential Integration

$$\int \frac{du}{u} = \ln|u| + c$$

$$\int \frac{f'(x)dx}{f(x)} = \ln|f(x)| + c$$

$$\int \tan u \, du = -\ln|\cos u| + c$$

$$\int \cot u \, du = \ln|\sin u| + c$$

$$\int \sec u \, du = \ln|\sec u + \tan u| + c$$

$$\int \csc u \, du = -\ln|\csc u + \cot u| + c$$

$$\int e^x dx = e^x + c$$

$$\int a^x dx = \frac{a^x}{\ln a} + c$$

5.9 Inverse Trig Functions

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\frac{u}{a} + c$$

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\arctan\frac{u}{a} + c$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\operatorname{arcsec}\frac{|u|}{a} + c$$

6.2-6.4 – Volume, Arc Length and Surface Area

Arc length $= \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$

Volume $= \int_a^b \pi r^2 dx$

Surface area $= \int_a^b 2\pi r\sqrt{1 + [f'(x)]^2}\, dx$