3.1 – Number Operations and Equality

Algebraic Postulates of Equality:

Reflexive Property: a=a (Any number is equal to itself.)

Substitution Property: If a=b, then a can be substituted for b in any expression.

Addition Property: If a=b, then a+c=b+c

Subtraction Property: If a=b, then a-c=b-c.

Multiplication Property: If a=b, then ac=bc.

Division Property: If a=b, then a/c=b/c.

State the property of equality illustrated by each statement:

3. If c/d=π, then c=πd

Multiplication Property of equality

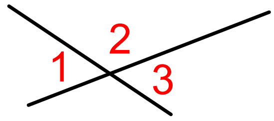
4. If ∠A+∠B+∠C=180° and ∠C=∠A+∠B, then ∠C+∠C=180°.

Substitution Property of Equality

5. If 2∠C=180°, then ∠C=90°.

Division Property of Equality

This figure shows two lines intersecting to form several angles, three of which are numbered.



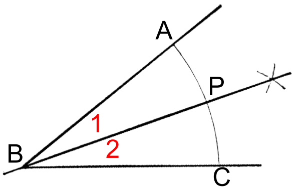
8. If ∠1+∠2=∠2+∠3, then ∠1=∠3. Why?

Subtraction Property of Equality (by ∠2)

9. If ∠1=∠2 and ∠2=∠3, then ∠1=∠3. Why?

Substitution Property of Equality

This figure shows how we bisected an angle by using a straightedge and compass. Let's check the algebra to see that ∠1 is the size that we would expect.



11. If ∠ABC=∠1+∠2 and ∠1=∠2, then ∠ABC=∠1+∠1=2∠1. Why?

Substitution Property of Equality

12. If ∠ABC=2∠1, then ∠1=(1/2) ∠ABC. Why?

Multiplication by ½
Division by 2

Quadratic formula

If $ax^2 + bx + c = 0$, then $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$.

37. What is the hypothesis of this theorem?

$ax^2+bx+c=0$

Name the postulate that is the reason for each of the following first three steps in its proof:

38. If $ax^2 + bx + c = 0$, then $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$.

Division Property of Equality (by a)

39. If $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$, then $x^2 + \frac{b}{a}x = -\frac{c}{a}$.

Subtraction Property of Equality (by $\frac{c}{a}$)

40. If $x^2 + \frac{b}{a}x = -\frac{c}{a}$, then $x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$.

Addition Property of Equality (by $\left(\frac{b}{2a}\right)^2$)

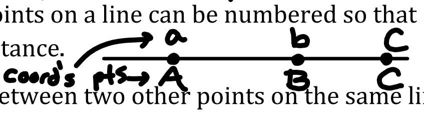
41. What kind of proof begins like this?

Direct (syllogism)

3.2 – The Ruler and Distance

distance between a & b is $|a-b| = |b-a|$

Postulate 3: The Ruler Postulate – The points on a line can be numbered so that positive number differences measure distance.



Def: Betweenness of Points – A point is between two other points on the same line iff its coordinate is between their coordinates.
(More briefly, A-B-C iff a<b<c or a>b>c.)

$a < b < c$

$a < b$ and $b < c$

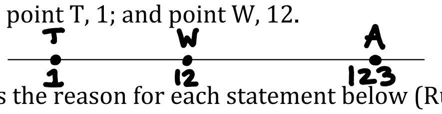
Theorem 1: The Betweenness of Points Theorem

If A-B-C, then AB+BC=AC

Proof for a<b<c case:

Statements:	Reasons:
TG A-B-C	The hypothesis.
ʹG a<b<c	Definition of betweenness.
,G AB=b-a and BC=c-b	Ruler Postulate.
3.5 AB+BC=b-a+BC	Addition
ИG AB+BC=(b-a)+(c-b)=c-a	~~Addition~~ Substitution (and simplifaction).
иG AC=c-a	Ruler Postulate.
G AB+BC=AC	Substitution (steps 4 and 5).

Three points on a line have the following coordinates:
point A, 123; point T, 1; and point W, 12.



Which idea is the reason for each statement below (Ruler Postulate, definition of betweenness of points, or Betweenness of Points Theorem)?

4. T-W-A because 1<12<123.

Definition of betweenness of points

5. TW+WA=TA because T-W-A.

Betweenness of points Theorem

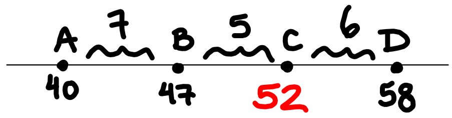
Suppose point A is at coordinate 40, point B is at coordinate 47, distance BC is 5, and point D is at coordinate 58. Determine:

TG The total distance AD.

58-40 = 18

'G The coordinate of C.

47+5 = 52



,G The distance CD.

58-52 = 6

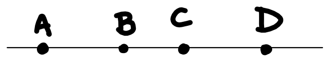
Because A-B-C, AB+BC=AC, or 7+5=12, according to the Betweenness of Points Theorem. Use this theorem to complete the statements:

TG 9. Because B-C-D, BC+ CD = BD , or 5+ 6 = 11 .

'G 10. Because A-B-D, AB+ BD = AD , or 7+ 11 = 18 .

,G 11. Because A-C-D, AC+ CD = AD , or 12 + 6 = 18 .

Suppose AC=BD. Complete the statements:



38. Because A-B-C, AC=AB +BC

by the Betweenness of Points Theorem

39. Because B-C-D, BD= BC+CD

by the Betweenness of Points Theorem

40. Why is AB+BC=BC+CD?

Substitution (AC=BD given subst. into 38&39)

41. Why is AB=CD?

Subtraction Property of Equality (by BC)

3.3 – The Protractor and Angle Measure

Postulate 4: The Protractor Postulate – The rays in a half-rotation can be numbered from 0 to 180 so that positive number differences measure angles.

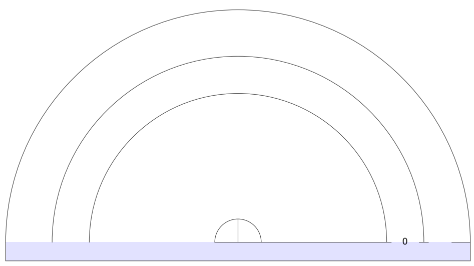
Definitions: An angle is

Acute iff it is less than 90°.

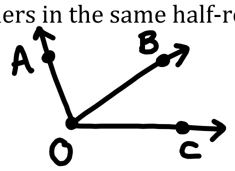
Right iff it is 90°.

Obtuse iff it is more than 90° but less than 180°.

Straight iff it is 180°.



Def: Betweenness of Rays – A ray is between two others in the same half-rotation iff its coordinate is between their coordinates.
(More briefly, OA-OB-OC iff a<b<c or a>b>c.)



Theorem 2: The Betweenness of Rays Theorem –
If OA-OB-OC, then ∠AOB+∠BOC=∠AOC.

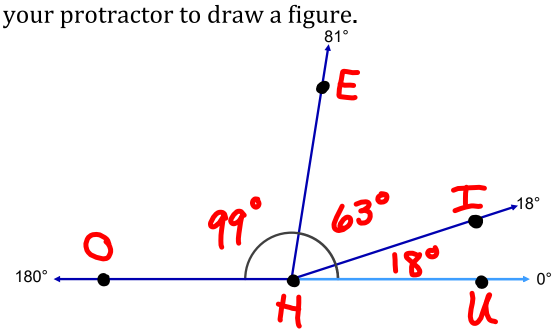
Proof for a>b>c case:

Statements:		**Reasons:**
TG	OA-OB-OC	The hypothesis.
ʹG	a>b>c	Definition of betweenness.
ˏG	∠AOB=a-b and ∠BOC=b-c	Protractor Postulate.
ИG	∠AOB+∠BOC=(a-b)+(b-c)=a-c	Addition (and simplification).
иG	∠AOC=a-c	Protractor Postulate.
G	∠AOB+∠BOC=∠AOC	Substitution (steps 4 and 5).

Three rays in a half-rotation have the following coordinates:
ray HE, 81; ray HI, 18; and ray HO, 180.

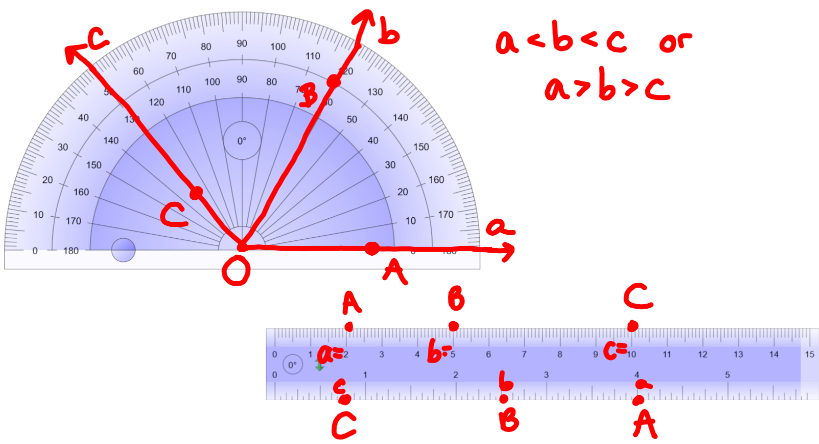
4. Which ray is between the other two (and why)?

Use your protractor to draw a figure.



5. Name and find the measures of the three angles formed by the rays.

m∠UHI = 18°
m∠IHE = 81-18 = 63°
m∠EHO = 180-81 = 99°



3.4 – Bisection

Def: A point is on the midpoint of a line segment iff it divides the line segment into two equal segments.

Def: A line bisects an angle iff it divides the angle into two equal angles.

Def: Two objects are congruent if and only if they coincide exactly when superimposed.

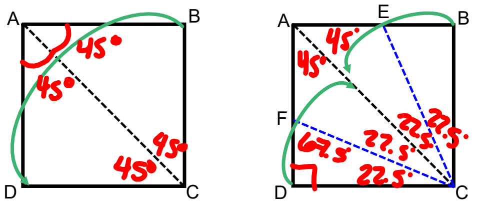
≅

Def: A corollary is a theorem that can be easily proved as a consequence of a postulate or another theorem.

Corollary to the Ruler Postulate: A line segment has exactly one midpoint.
(unique mid point)

Corollary to the Protractor Postulate: An angle has exactly one ray that bisects it.
(unique angle bisector)

Bisecting angles with origami: Starting with a square sheet of paper, corner B is folded onto D. Then sides BC and DC are folded onto the fold AC.



Because ∠BAC fits onto ∠DAC, ∠BAC and ∠DAC are congruent.

17. Which angle is bisected if ∠BAC=∠DAC?

∠BAD = ∠DAB

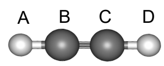
18. Name three more angles that are bisected in the folding process.

∠DCB, ∠ACB , ∠DCA

Angle BCD is a right angle because the process starts with a square. Find the number of degrees in each of the following angles.

20. ∠FCD = 22.5°   21. ∠FCE = 45°   23. ∠DFC = 67.5°

Acetylene molecules contain four atoms, arranged linearly.



34. In this molecule, AB=CD, A-B-C and B-C-D. Use these facts to supply the reasons in the following direct proof that AC=BD.

Proof:

Statements	Reasons
1. AB=CD	Given.
2. AB+BC=BC+CD	Addition (by BC)
3. A-B-C and B-C-D	Given
4. AB+BC=AC and BC+CD=BD	Betweenness of Points Theorem
5. Therefore, AC=BD	Substitution (#4 into #2)

35. Use the additional fact that AC>2AB to supply the missing statements and reasons in this indirect proof that B is *not* the midpoint of AC.

Proof:

Statements	Reasons
Suppose B is the midpoint of AC	Assumption
If B is the midpoint of AC, then AB=BC.	Def. of midpoint = midpoint divides segment into two equal segment
Because AB+BC=AC, 2AB=AC.	Substitution (BC for AB) & simplification
This contradicts that AC > 2AB	Hypothesis
Therefore, our assumption is false and B is not the midpoint of AC.	

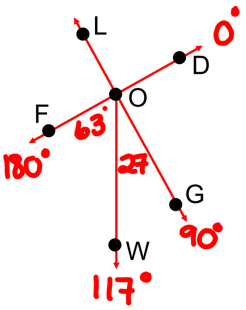
3.5 – Complementary and Supplementary Angles

Def: Two angles are complementary iff their sum is 90°.

Def: Two angles are supplementary iff their sum is 180°.

Theorem 3: Complements of the same angle are equal. (proved on p.106)

Theorem 4: Supplements of the same angle are equal.



If a protractor is placed on the figure so that OD has coordinate 0, the coordinates of the other rays are: OG, 90; OW, 117; OF, 180.

16. Write the equation that follows from the fact that OD-OW-OF. ∠DOW + ∠WOF = ∠DOF (Betweenness of Rays Theorem)

17. Find the measures of ∠DOW = 117°

∠WOF = 63°

∠DOF = 180°

18. What relation does ∠DOW have to ∠WOF?

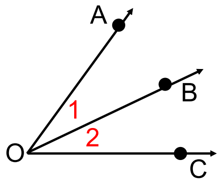
they are supplementary ∠'s

19. Find the measure of ∠WOG

27°

20. What relation does ∠WOG have to ∠WOF?

they are complementary angles



$\angle 1 + \angle AOC = 90^\circ$ } def. of complementary ∠'s
$\angle 2 + \angle AOC = 90^\circ$
$\angle 1 + \angle AOC = \angle 2 + \angle AOC$ (substitution)
$\angle 1 = \angle 2$ (subtraction)

In the figure, ∠1 and ∠2 are both complements of ∠AOC.

44. What else is true?

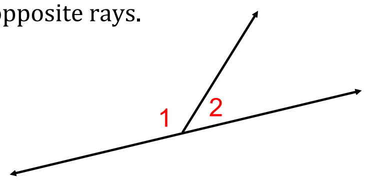
$\angle 1 = \angle 2$ (complements of the same angle are equal)

45. Is it possible to figure out the size of each angle in the figure without measuring them?

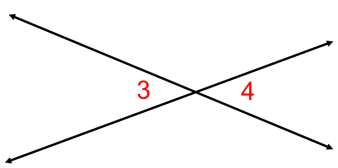
$\angle 1 + \angle 2 = \angle AOC$ (Betweenness of Rays Theorem)
$\angle 2 + \angle 2 = \angle AOC$ Substitution
$\angle 2 + \angle 2 + \angle 2 = \angle AOC + \angle 2$ Addition (by ∠2)
$3\angle 2 = 90^\circ$ Substitution $90^\circ = \angle AOC + \angle 2$
$\angle 2 = 30^\circ$ Division
$\angle 1 = 30^\circ$ substitution
$\angle AOC = 60^\circ$ substitution

3.6 – Linear Pairs and Vertical Angles

Def: Two angles are a linear pair iff they have a common side and their other sides are opposite rays.



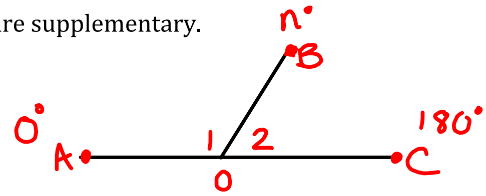
Def: Two angles are vertical angles iff the sides of one angle are opposite rays to the sides of the other.



Theorem 5: The angles in a linear pair are supplementary.

Given: ∠1 and ∠2 are a linear pair.

Prove: ∠1 and ∠2 are supplementary.



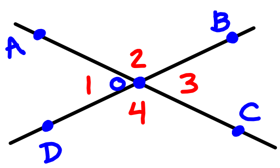
Proof:

Statements	Reasons
1. ∠1 and ∠2 are a linear pair.	Given
2. Rays OA and OC are opposite rays.	If two angles are a linear pair, they have a common side and their other sides are opposite rays. (Definition of Linear Pair)
3. Let the coordinates of OA, OB, and OC be 0, n, and 180.	Protractor Postulate
4. ∠1=n-0=n° and ∠2=(180-n)°	Protractor Postulate
4.5 ∠1+∠2 = n° +∠2	
5. ∠1+∠2= n°+(180-n)°= 180°	Addition → substitution ∠2=180-n°
6. ∠1 and ∠2 are supplementary.	Two angles are supplementary if their sum is 180°.

Theorem 6: Vertical angles are equal.

Given: ∠1 and ∠3 are vertical angles.

To Prove: ∠1=∠3



Proof

Statements	Reasons
1. ∠1 and ∠3 are vertical angles.	Given
2. OA and OC are opposite Rays OB and OD are opposite Rays	Def. of vertical angles
3. ∠1 and ∠2 form a linear pair ∠2 and ∠3 form a linear pair	Definition of linear pair
4. ∠1 and ∠2 are supplementary ∠2 and ∠3 are supplementary	Angles in a linear pair are supplementary
5. ∠1=∠3	Supplements of the same angle are equal.

3.7 – Perpendicular and Parallel Lines

if and only if

Def: Two lines are perpendicular iff they form a right angle.

Theorem 7: Perpendicular lines form four right angles.

Corollary to the definition of a right angle: All right angles are equal.

Theorem 8: If the angles in a linear pair are equal, then their sides are perpendicular.

Def: Two lines are parallel iff they lie in the same plane and do not intersect.