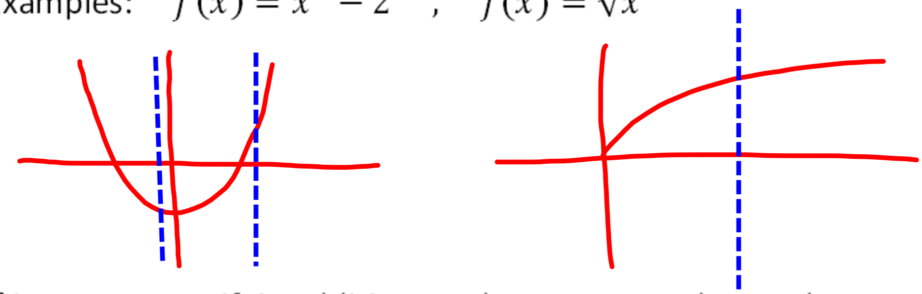
Inverse Trigonometric Functions

Recall from Algebra:

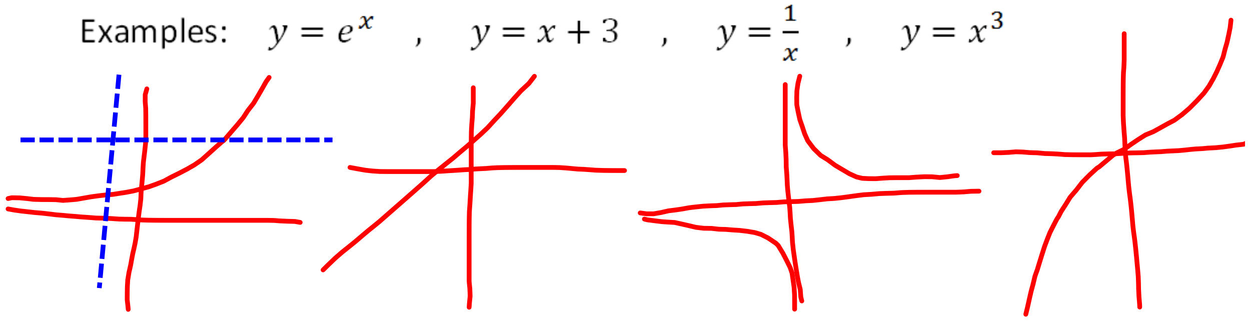
- f is a **function** if each input value (x) has a unique output $f(x)$.

  Examples: $f(x) = x^2 - 2$, $f(x) = \sqrt{x}$



- f is **one-to-one** if, in addition, each y corresponds to only one x.

  Examples: $y = e^x$, $y = x + 3$, $y = \frac{1}{x}$, $y = x^3$



- If f is a one-to-one function, we can define its inverse $f^{-1}(x)$.

  Note that this notation is not exponentiation, i.e. $f^{-1}(x) \neq \frac{1}{f(x)}$

$x^{-n} = \frac{1}{x^n}$

- $f(x)$ and $g(x)$ are **inverses** if

  $(f \circ g)(x) = f\big(g(x)\big) = x = g\big(f(x)\big) = (g \circ f)(x)$,

  that is, **inverse functions "undo" each other**.

Example: $f(x) = x^3$, $g(x) = \sqrt[3]{x}$

$(f \circ g)(x) = (\sqrt[3]{x})^3 = x$

$(g \circ f)(x) = \sqrt[3]{x^3} = x$

What do we mean by an Inverse Trig function?

Recall that **for a basic Trigonometric function**, e.g. $f(x) = \sin x$,

- **The input (x) is an angle**
- **The output $f(x)$ is a ratio of sides**

So **for an inverse Trigonometric function,**

- **The input (x) is a ratio of sides**
- **The output $f(x)$ is an angle**

Construction of the inverse of $f(x) = \sin x$:

$f(x) = x^3 - 8$

$y = x^3 - 8$

$x = y^3 - 8$

$x + 8 = y^3$

$\sqrt[3]{x+8} = y$

$\sqrt[3]{x+8} = f^{-1}(x)$

$y = \sin x$

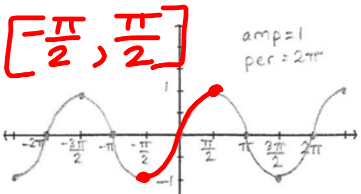
$x = \sin y$

$y =$ the angle whose sine value is x

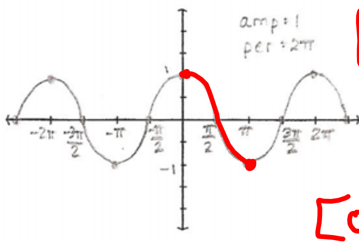
$y = \sin^{-1} x = \arcsin x$

But Trigonometric functions aren't one-to-one – how is the inverse defined? We must restrict the domain!

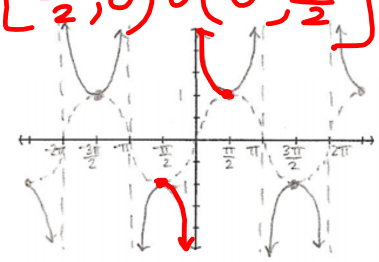
$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



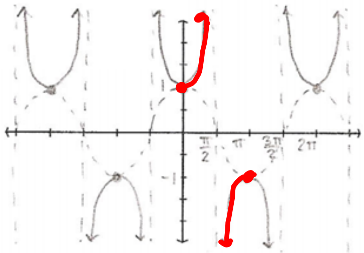
$[0, \pi]$

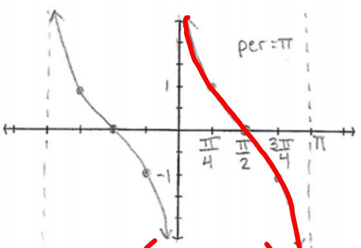


$\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

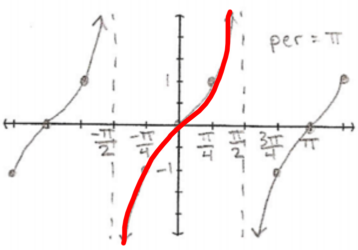


$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$





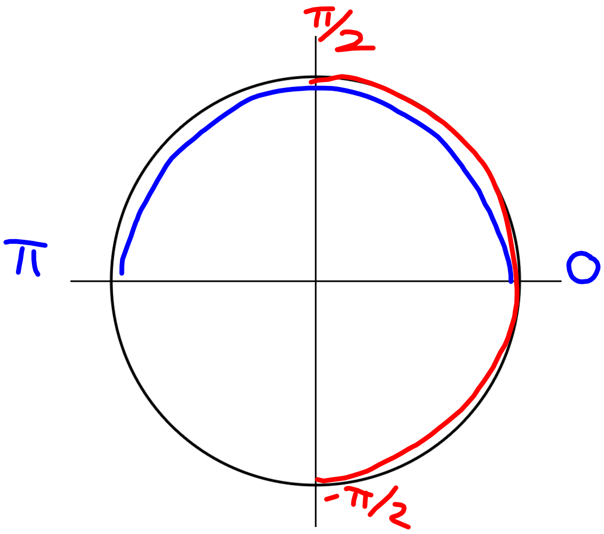
$(0, \pi)$



$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

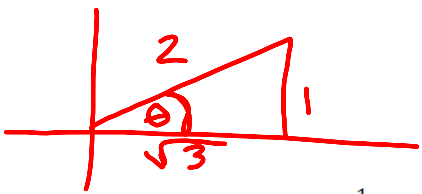
Summary of Restricted Domains:

Interval	Functions	Quadrants
$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$\sin x, \csc x, \tan x$	IV & I
$(0, \pi)$	$\cos x, \sec x, \cot x$	I & II



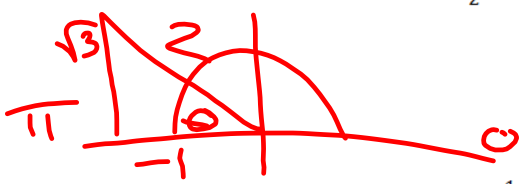
Evaluate the inverse trigonometric expression.

$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$



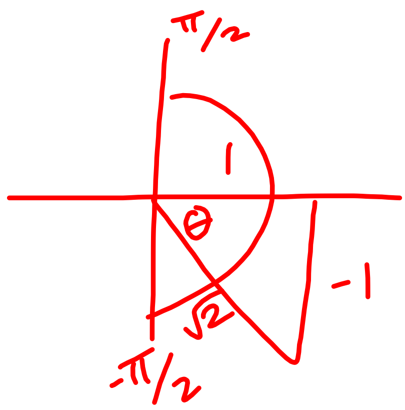
In words: What angle θ, between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ (the restricted domain for sine) is such that $\sin\theta = \frac{1}{2}$?

$\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$



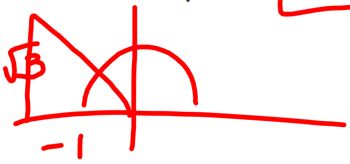
In words: What angle θ, between 0 and π (the restricted domain for cosine) is such that $\cos\theta = -\frac{1}{2}$?

$\tan^{-1}(-1) = -\frac{\pi}{4}$



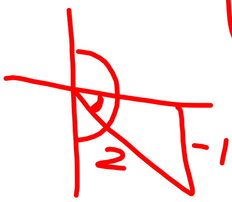
Evaluate.

1. $\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right) = \boxed{\frac{2\pi}{3}}$

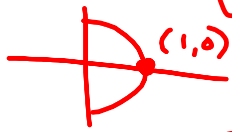


2. $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \boxed{\frac{\pi}{3}}$

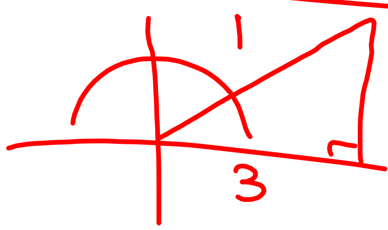
3. $\csc^{-1}(-2) = \boxed{-\frac{\pi}{6}}$



4. $\tan^{-1}(0) = \boxed{0}$



5. $\cos^{-1}(3) = \boxed{\text{undefined}}$



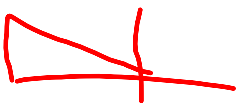
What happens when we compose a Trigonometric function with its inverse?

According to the definition,

$f(x)$ and $g(x)$ are inverses if $f\big(g(x)\big) = x$ and $g\big(f(x)\big) = x$
(for all x-values in the respective domains of g and f)

We would then expect

$\sin(\sin^{-1} x) = x$ and $\sin^{-1}(\sin x) = x$



$\sin\left(\sin^{-1}\frac{1}{2}\right) = \frac{1}{2}$

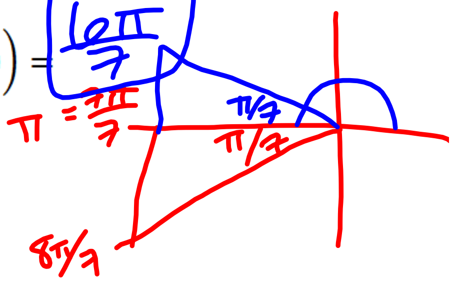
$\sin(30^\circ) =$

$\sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right) = \sin^{-1}\left(\frac{1}{2}\right) = \boxed{\frac{\pi}{6}}$

$\sin^{-1}\left(\sin\left(-\frac{\pi}{6}\right)\right) = -\frac{\pi}{6}$

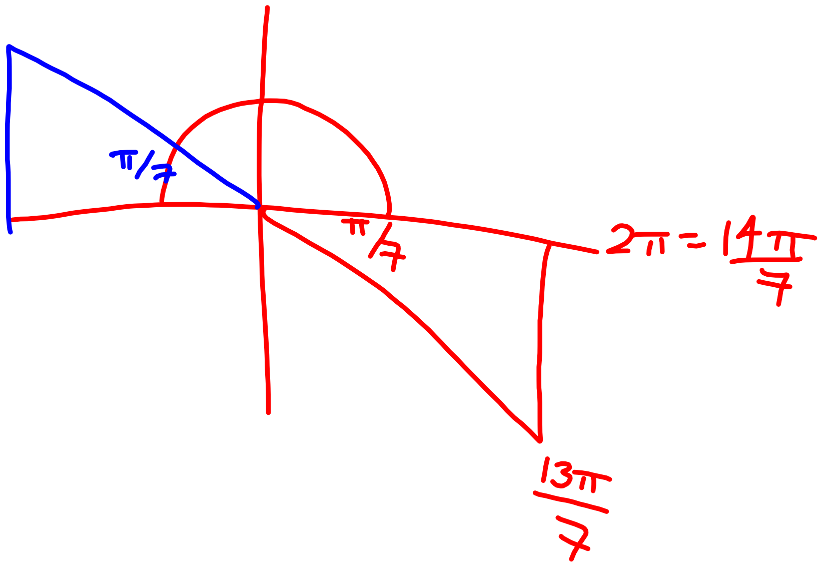
$\sin^{-1}\left(-\frac{1}{2}\right) =$

$\cos^{-1}\left(\cos\left(\frac{8\pi}{7}\right)\right) = \boxed{\frac{6\pi}{7}}$



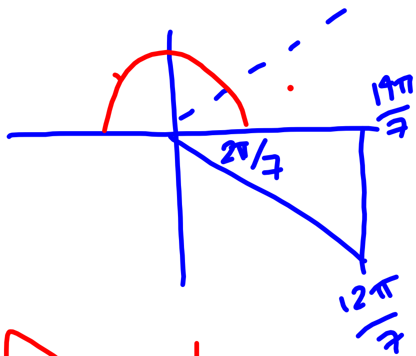
$\sin(\sin^{-1} 3) =$ undefined

$$\cot^{-1}\left(\cot\frac{13\pi}{7}\right) = \boxed{\frac{6\pi}{7}}$$

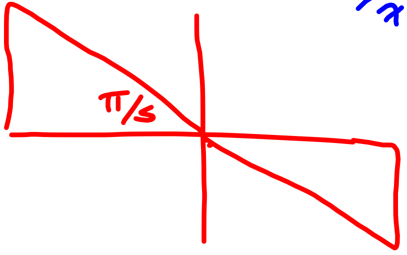


Evaluate:

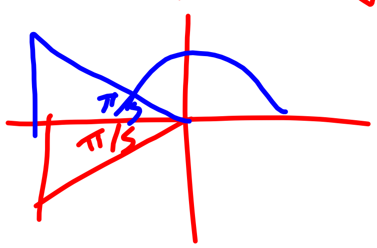
$$\cos^{-1}\left(\cos\left(\frac{12\pi}{7}\right)\right) = \boxed{\frac{2\pi}{7}}$$



$$\tan^{-1}\left(\tan\left(\frac{4\pi}{5}\right)\right) = \boxed{\frac{-\pi}{5}}$$



$$\sec^{-1}\left(\sec\left(-\frac{4\pi}{5}\right)\right) = \boxed{\frac{4\pi}{5}}$$



HW #7 (due Fri, 09/26)

- 6.1#1-69 odd proofs
- 6.3 #1-24 all double- and half-angle identities (application & proof)
  #30-36 all
  #49-93 odd
- 6.5 #1-24 all inverse functions
- 6.6 solving trig equations