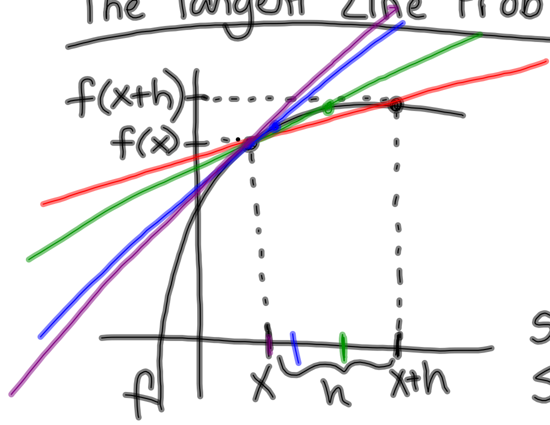
2.1 The Derivative & The Tangent Line Problem



secant line
crosses through a function at two points

slope of the secant line:

$$\frac{f(x+h)-f(x)}{x+h-x} = \frac{f(x+h)-f(x)}{h}$$

"the difference quotient"

what happens as $h \to 0$?

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

As $h \to 0$, the secant line approximates the tangent line, and the limit is the slope of the tangent line and we call it the derivative of f at x.

$f'(x)$ "f prime of x"

$\frac{dy}{dx}$, y', $\frac{d}{dx}[f(x)]$, $D_x[y]$

"derivative of y with respect to x"
"y prime"
"the derivative w.r.t. x of f(x)"
"partial derivative w.r.t. x of y"

2.1

22. $f(x) = \frac{1}{x^2}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{x^2}{x^2}\cdot\frac{1}{(x+h)^2} - \frac{1}{x^2}\cdot\frac{(x+h)^2}{(x+h)^2}}{\frac{h}{1}} = \lim_{h\to 0} \frac{x^2-(x+h)^2}{x^2(x+h)^2}\cdot\frac{1}{h}$$

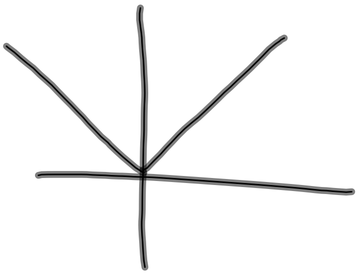
$$= \lim_{h\to 0} \frac{\cancel{x^2} - \cancel{x^2} - 2xh - h^2}{hx^2(x+h)^2} = \lim_{h\to 0} \frac{\cancel{h}(-2x-h)}{\cancel{h}x^2(x+h)^2} = \lim_{h\to 0} \frac{(-2x-h)}{x^2(x+h)^2}$$

$$= \frac{-2x}{x^4} = \boxed{\frac{-2}{x^3}}$$

Differentiability & Continuity

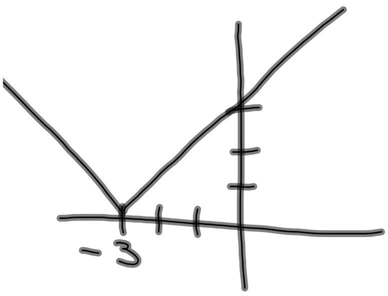
$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

All differentiable functions are continuous.
Not all continuous functions are differentiable.



sharp points hurt
the calculus cockroach

$f(x) = |x+3|$; $x = -3$



$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = \lim_{x \to -3^-} \frac{|x+3| - |-3+3|}{x - (-3)}$$

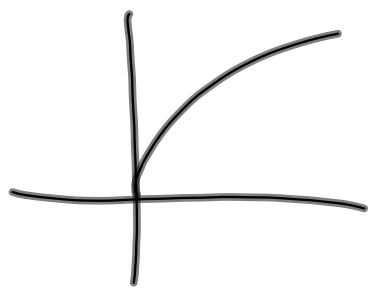
$$= \lim_{x \to -3^-} \frac{|x+3|}{x+3} = -1$$

$$\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} = \lim_{x \to -3^+} \frac{|x+3|}{x+3} = 1$$

$$\frac{|x+3|}{x+3} = \begin{cases} \frac{x+3}{x+3} = 1, & x+3 > 0,\ x > -3 \\ \frac{-(x+3)}{x+3} = -1, & x+3 < 0,\ x < -3 \end{cases}$$

Since left- & right-hand limits are different, the limit does not exist. Therefore $f(x) = |x+3|$ is not differentiable at $x = -3$

$f(x) = \sqrt{x}$



$$\lim_{x \to 0^+} \frac{f(x) - f(c)}{x - 0}$$

$$= \lim_{x \to 0^+} \frac{\sqrt{x} - \sqrt{0}}{x - 0} = \lim_{x \to 0^+} \frac{\sqrt{x}}{x} = \lim_{x \to 0^+} \frac{x^{1/2}}{x^1}$$

$$= \lim_{x \to 0^+} \frac{1}{x^{1/2}} = \lim_{x \to 0^+} \frac{1}{\sqrt{x}} = \infty$$

$\Rightarrow$ vertical tangent line

derivative is undefined. (not differentiable)

2.2 Basic Differentiation Rules

- the derivative of a constant function is 0.

for $c \in \mathbb{R}$, $\frac{d}{dx}[c] = 0$. $c' = 0$

<u>Proof</u> for $f(x) = c$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} =$$

$$\lim_{h \to 0} \frac{0}{h} = \lim_{h \to 0} 0 = 0.$$

• the power rule

for $n \in \mathbb{Q}$, $\frac{d}{dx}[x^n] = nx^{n-1}$

Recall binomial expansion:

$(x+h)^n = x^n + nx^{n-1}h + \frac{n(n-1)x^{n-2}}{2} \cdot h^2 + \ldots + h^n$

Proof

$\frac{d}{dx}[x^n] = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h \to 0} nx^{n-1} + \frac{n(n-1)x^{n-2}h}{2} + \ldots$

0

$= \boxed{nx^{n-1}}$

special case:

$\frac{d}{dx}[x] = 1 \cdot x^0 = 1 \cdot 1 = \boxed{1}$

$$\frac{d}{dx}\left[x^7\right] = 7x^6$$

$$\frac{d}{dx}\left[\pi^3\right] = 0$$

$$\frac{d}{dx}(2e) = 0$$

$$\frac{d}{dx}\left[\sqrt{x}\right] = \frac{d}{dx}\left[x^{1/2}\right] = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}\left[\frac{1}{x^3}\right] = \frac{d}{dx}\left[x^{-3}\right] = -3x^{-4} = \frac{-3}{x^4}$$

HW

2.1 #21, 23

2.2 #1-10