1.3

59. $\lim_{\Delta x\to 0} \frac{2(x+\Delta x)-2x}{\Delta x} = \lim_{\Delta x\to 0} \frac{2x+2\Delta x-2x}{\Delta x} =$

$= \lim_{\Delta x\to 0} \frac{2\cancel{\Delta x}}{\cancel{\Delta x}} = \lim_{\Delta x\to 0} 2 = \boxed{2}$

61. $\lim_{\Delta x\to 0} \frac{(x+\Delta x)^2-2(x+\Delta x)+1-(x^2-2x+1)}{\Delta x}$

$= \lim_{\Delta x\to 0} \frac{\cancel{x^2}+2x\Delta x+(\Delta x)^2-\cancel{2x}-2\Delta x+\cancel{1}-\cancel{x^2}+\cancel{2x}-\cancel{1}}{\Delta x}$

$= \lim_{\Delta x\to 0} \frac{\cancel{\Delta x}(2x+\Delta x-2)}{\cancel{\Delta x}} = \boxed{2x-2}$

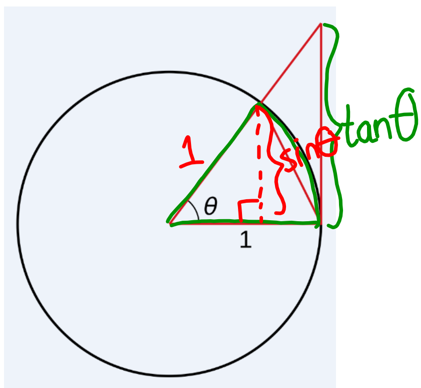
1.3 The Squeeze Theorem

$$\lim_{x\to 0}\frac{\sin x}{x} = ?$$

Area of whole circle $= \pi r^2|_{r=1} = \pi$

$$\frac{\text{Area of whole circle}}{\text{Total angle of circle}} = \frac{\text{Area of sector}}{\theta}$$

$$\frac{\pi}{2\pi} = \frac{\text{Area of sector}}{\theta} \rightarrow \text{Area of sector} = \frac{\theta}{2}$$



Area of outer triangle ≥ Area of sector ≥ Area of inner triangle

$$\frac{\tan\theta}{2} \geq \frac{\theta}{2} \geq \frac{\sin\theta}{2}$$

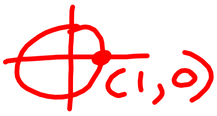
Multiply through by $\frac{2}{\sin\theta}$

$$\frac{\sin\theta}{2\cos\theta}\cdot\frac{2}{\sin\theta} \geq \frac{\theta}{2}\cdot\frac{2}{\sin\theta} \geq \frac{\sin\theta}{2}\cdot\frac{2}{\sin\theta}$$

$$\frac{1}{\cos\theta} \geq \frac{\theta}{\sin\theta} \geq 1$$

Take reciprocals and reverse inequalities

$$\cos\theta \leq \frac{\sin\theta}{\theta} \leq 1$$

Take limits 

$$\lim_{\theta \to 0} \cos\theta \quad \leq \quad \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \quad \leq \quad \lim_{\theta \to 0} 1$$

$$1 \leq \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \leq 1$$

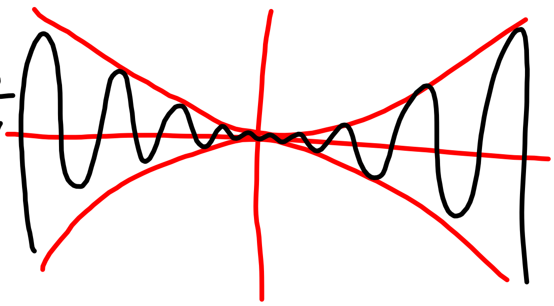
$$\Rightarrow \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

The Squeeze Theorem:

If $f(x) \leq g(x) \leq h(x)$ *and* $\lim_{x \to c} f(x) = L = \lim_{x \to c} h(x)$,

Then $\lim_{x \to c} g(x) = L$.

$$-x^2 \leq x^2 \sin x \leq x^2$$



Special Limits Derived by Squeeze Theorem:

$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1} \; ; \; \boxed{\lim_{x \to 0} \frac{1 - \cos x}{x} = 0}$$

Memorize!!

Use the squeeze theorem to find

$$\lim_{x\to 0}\left(x^2 \cos\frac{5}{x} - 3\right)$$

$$-1 \le \cos\boxed{\frac{5}{x}} \le 1$$

$$-x^2 \le x^2\cos\frac{5}{x} \le x^2$$

$$-x^2-3 \le x^2\cos\frac{5}{x}-3 \le x^2-3$$

$$\lim_{x\to 0}(-x^2-3) \le \lim_{x\to 0}\left(x^2\cos\frac{5}{x}-3\right) \le \lim_{x\to 0}(x^2-3)$$

$$-3 \le \lim_{x\to 0}\left(x^2\cos\frac{5}{x}-3\right) \le -3$$

$$\Rightarrow \lim_{x\to 0}\left(x^2\cos\frac{5}{x}-3\right) = \boxed{-3}$$

68. $\lim_{x\to 0}\frac{3(1-\cos x)}{x}$

$$= \left(\lim_{x\to 0}(3)\right)\cdot\left(\lim_{x\to 0}\frac{1-\cos x}{x}\right)$$

$$= 3 \cdot 0$$

$$= \boxed{0}$$

72. $\lim\limits_{x\to 0} \dfrac{\tan^2 x}{x}$ $= \lim\limits_{x\to 0} \dfrac{\sin^2 x}{x\cos^2 x}$

$= \lim\limits_{x\to 0} \left(\dfrac{\sin x}{x}\right)\left(\dfrac{\sin x}{\cos^2 x}\right)$

$= \left(\lim\limits_{x\to 0} \dfrac{\sin x}{x}\right)\left(\lim\limits_{x\to 0} \dfrac{\sin x}{\cos^2 x}\right)$

$= 1 \cdot 0$

$= \boxed{0}$

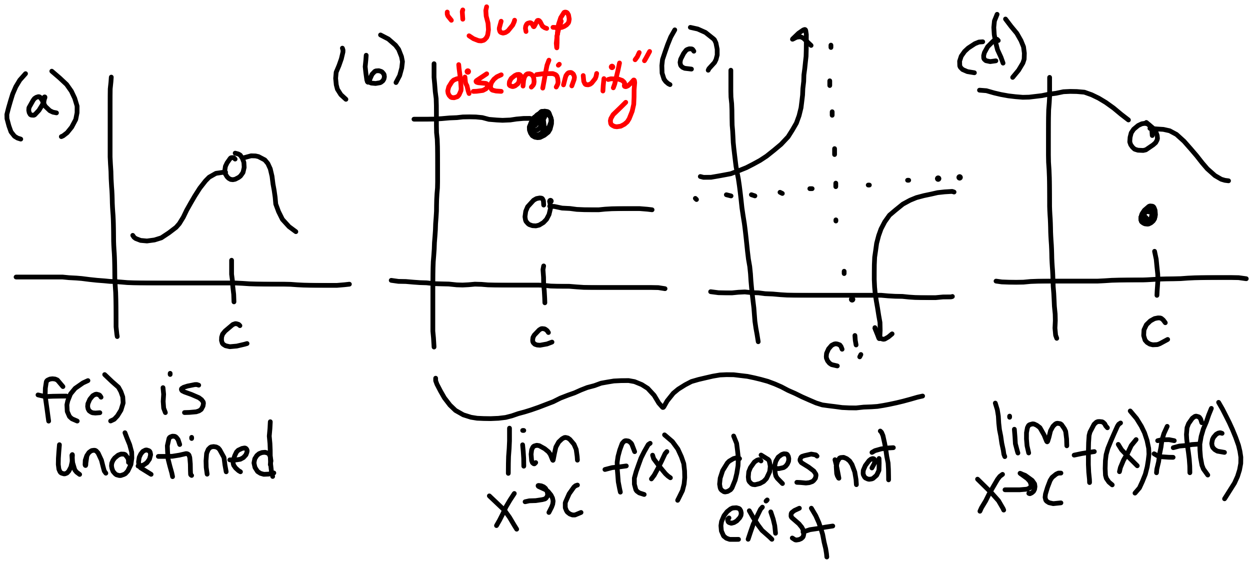
78. $\lim\limits_{x\to 0} \dfrac{\sin 2x}{\sin 3x}$ $= \lim\limits_{x\to 0} \dfrac{\frac{\sin 2x}{2x} \cdot 2}{\frac{\sin 3x}{3x} \cdot 3}$

$\dfrac{\frac{1}{2x} \cdot 2}{\frac{1}{3x} \cdot 3} = \dfrac{x}{x} = 1$

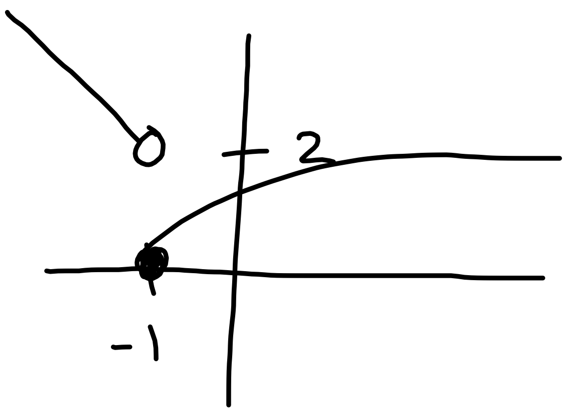
$= \dfrac{\underbrace{\lim\limits_{x\to 0}\left(\frac{\sin 2x}{2x}\right)}_{1} \cdot \underbrace{\lim\limits_{x\to 0} 2}_{2}}{\underbrace{\lim\limits_{x\to 0}\left(\frac{\sin 3x}{3x}\right)}_{1} \cdot \underbrace{\lim\limits_{x\to 0} 3}_{3}}$

$= \boxed{\dfrac{2}{3}}$

1.4 Connuity and One-Sided Limits



These are all discontinuities
(a) and (d) are removable
(b) and (c) are nonremovable



$$\lim_{x\to -1^-} f(x) = 2$$

$$\lim_{x\to -1^+} f(x) = 0$$

$$\lim_{x\to -1} f(x) = \text{does not exist}$$

One-Sided Limits

$\lim_{x\to c^+} f(x) = L$ *limit from the right*

$\lim_{x\to c^-} f(x) = L$ *limit from the left*

$$\boxed{\lim_{x\to c} f(x) = L \ \text{if and only if} \ \lim_{x\to c^-} f(x) = L = \lim_{x\to c^+} f(x)}$$

Continuity at a point

A function f is continuous at c if the following 3 conditions are met:

1. $f(c)$ is defined
2. Limit of $f(x)$ exists when x approaches c
3. Limit of $f(x)$ when x approaches c is equal to $f(c)$

f(x) is continuous at c if

$$\lim_{x \to c} f(x) = f(c)$$

Continuity on an open interval

A function is continuous on an open interval if it is continuous at each point in the interval. A function that is continuous on the entire real line $(-\infty, \infty)$is everywhere continuous.

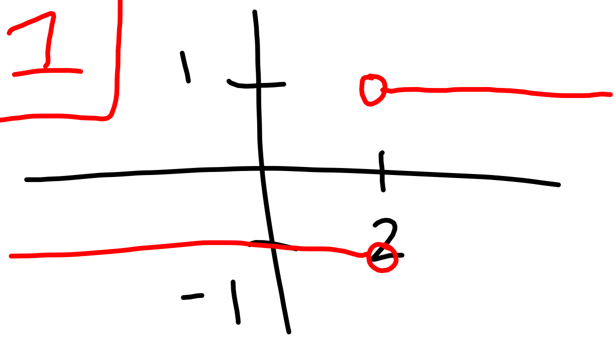
Continuity on a closed interval

A function f is continuous on the closed interval $[a, b]$ if it is continuous on the open interval I (a, b) and $\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$.

10. $$\lim_{x \to 4^-} \frac{\sqrt{x} - 2}{x - 4} \cdot \frac{\sqrt{x} + 2}{\sqrt{x} + 2}$$

$$= \lim_{x \to 4^-} \frac{\cancel{x-4}\;1}{\cancel{(x-4)}(\sqrt{x}+2)} = \boxed{\frac{1}{4}}$$

12. $\lim\limits_{x\to 2^+} \dfrac{|x-2|}{x-2} = \boxed{1}$



$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$\frac{|x-2|}{x-2} = \begin{cases} \dfrac{x-2}{x-2} = 1, & x-2 > 0 \\ & x > 2 \\ \dfrac{-(x-2)}{x-2} = -1, & x-2 < 0 \\ & x < 2 \end{cases}$$

1.3

67–77 odd; 87, 88