1. Find the limit L, and the value of $\delta > 0$ that guarantees the limit. Bonus points if you can prove that the limit is L with clear explanations (hint: begin by stating the $\varepsilon - \delta$ definition of the limit).

$$\lim_{x \to -2}(-5x-7) = -5(-2)-7 = 10-7 = \boxed{3 = L}$$

$$|f(x) - L| = |-5x-7-3| = |-5x-10| = |(-5)(x+2)| = 5|x+2|$$

$$5|x+2| < \varepsilon$$

$$|x+2| < \boxed{\varepsilon/5 = \delta}$$

Proof: Given $\varepsilon > 0$, take $\delta = \varepsilon/5$. Then whenever $|x-c| < \delta$ i.e. $|x+2| < \varepsilon/5$, we have $|f(x)-L| = \ldots = 5|x+2| < 5 \cdot \frac{\varepsilon}{5} = \varepsilon$ i.e. $|f(x)-L| < \varepsilon$. Hence $\lim_{x \to -2} f(x) = 3$.

2. Find the limit (if it exists).

$$\lim_{x \to -2} \frac{x^2+12x+20}{x^2-7x-18} = \lim_{x \to -2} \frac{(x+10)\cancel{(x+2)}}{(x-9)\cancel{(x+2)}} = \frac{-2+10}{-2-9} = \boxed{-\frac{8}{11}}$$

3. Find the limit (if it exists).

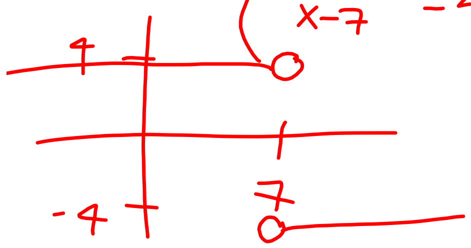
$$\lim_{x \to 11} \frac{9-\sqrt{x+70}}{11-x} \cdot \frac{9+\sqrt{x+70}}{9+\sqrt{x+70}} = \lim_{x \to 11} \frac{81-(x+70)}{(11-x)(9+\sqrt{x+70})}$$

$$= \lim_{x \to 11} \frac{\cancel{11-x}\ 1}{\cancel{(11-x)}(9+\sqrt{x+70})} = \frac{1}{9+\sqrt{11+70}} = \boxed{\frac{1}{18}}$$

4. Find the limit (if it exists). You must show work indicating how you found the limit.

$$\lim_{x \to 7^-} \frac{-4|x-7|}{x-7} = \boxed{4}$$

$$\frac{-4|x-7|}{x-7} = \begin{cases} \frac{-4(x-7)}{x-7} = -4, & x-7 > 0,\ x > 7 \\ \frac{(-4)[-(x-7)]}{x-7} = 4, & x-7 < 0,\ x < 7 \end{cases}$$



$$= (-4) \underbrace{\lim_{x \to 7^-} \frac{|x-7|}{x-7}}_{=-1}$$

5. Find the limit (if it exists). (Be careful about your negative signs!)

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\ ,\ \ where\ f(x)=5x^2-4$$

$$\lim_{h\to 0}\frac{5(x+h)^2-4-(5x^2-4)}{h}=\lim_{h\to 0}\frac{5(x^2+2xh+h^2)-4-5x^2+4}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{5x^2}+10xh+5h^2-\cancel{4}-\cancel{5x^2}+\cancel{4}}{h}=\lim_{h\to 0}\frac{\cancel{h}(10x+5h)}{\cancel{h}}$$

$$=\boxed{10x}$$

6. Find the limit (if it exists). You must show how you found the limit.

$$\lim_{x\to 0}\frac{\tan x}{2x}=\lim_{x\to 0}\frac{\sin x}{2x\cos x}=\left(\lim_{x\to 0}\frac{\sin x}{x}\right)\left(\lim_{x\to 0}\frac{1}{2\cos x}\right)$$

$$\downarrow \qquad \downarrow$$

$$1 \qquad 1/2$$

$$=\boxed{\frac{1}{2}}$$

7. Find the limit (if it exists).

$$\lim_{x\to 2^-} f(x)\ ,\qquad where\ f(x)=\begin{cases} x-7\ , & x\le -5 \\ x^2-6x+13\ , & -5<x\le 2 \\ -4\ , & x>2\end{cases}$$

$$2^2-6(2)+13=4-12+13=\boxed{5}$$

8. Find the limit by demonstrating use of the Squeeze Theorem. *Hint:* $-1\le \cos\theta\le 1$ *for all* θ.

$$\lim_{x\to 0}\left(\frac{1}{5}x^2\cos\frac{3}{x}-4\right)$$

$$-1\le\cos\frac{3}{x}\le 1$$

$$\lim_{x\to 0}\left(-\frac{1}{5}x^2-4\right)\le\lim_{x\to 0}\left(\frac{1}{5}x^2\cos\frac{3}{x}-4\right)\le\lim_{x\to 0}\left(\frac{1}{5}x^2-4\right)$$

$$\downarrow \qquad\qquad \downarrow$$

$$-4 \qquad\qquad -4$$

$$\Rightarrow \boxed{-4}$$

9. Determine if the Intermediate Value Theorem guarantees a c in the interval $[2,7]$ such that $f(c) = 8$. If so, find all such values of c.

$f(x) = x^2 - x - 12$

f is cts on $[2,7]$ ✓

$f(2) = 2^2 - 2 - 12 = -10 < 8$

$f(7) = 7^2 - 7 - 12 = 30 > 8$

$\Rightarrow$ IVT applies

$x^2 - x - 12 = 8$

$x^2 - x - 20 = 0$

$(x-5)(x+4) = 0$

$x = \boxed{5}$, $x = \cancel{-4}$

10. Discuss the continuity of the function. Identify all discontinuities (if any) as either removable or non-removable, and state the interval(s) on which the function is continuous.

$$f(x) = \frac{(x^2 + 5x - 36)(x - 12)}{(x - 4)(x^2 + 13x + 36)} = \frac{(x+9)(x-4)(x-12)}{(x-4)(x+9)(x+4)}$$

removable @ 4, -9

non-rem. @ -4

f is cts. on $(-\infty, -9) \cup (-9, -4) \cup (-4, 4) \cup (4, \infty)$

Bonus A:

Construct a function that has removable discontinuities at -9, -1, and 3, non-removable discontinuities in the form of vertical asymptotes at -2 and 4, and jump discontinuities at -6 and 2.

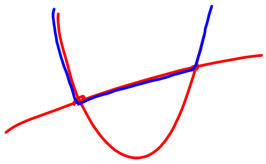
$$f(x) = \begin{cases} \dfrac{x+9}{x+9} & x < -6 \\ \dfrac{x+1}{(x+1)(x+2)} & -6 \le x < 2 \\ \dfrac{x-3}{(x-3)(x-4)} & x \ge 2 \end{cases}$$

Bonus B:

Determine the values of b and c such that the function is continuous on the entire real number line.

$$f(x) = \begin{cases} x - 1, & 1 < x < 5 \\ x^2 + bx + c, & |x - 3| \ge 2 \end{cases}$$

$$= \begin{cases} x^2 + bx + c & x \le 1 \\ x - 1 & 1 < x < 5 \\ x^2 + bx + c & x \ge 5 \end{cases}$$



$x = 1$

$1^2 + b(1) + c = 1 - 1$

$b + c = -1$

$c = -b - 1$

$= -(-5) - 1$

$= 5 - 1$

$\boxed{c = 4}$

$x = 5$

$5 - 1 = 5^2 + 5b + c$

$4 = 25 + 5b + c$

$5b + c = -21$

$-\ (b + c = -1)$

$4b = -20$

$\boxed{b = -5}$

2.2 Basic Differentiation Rules

1. The derivative of a constant function is zero, i.e.,

for $c \in \mathbb{R}$, $\frac{d}{dx}[c] = 0$

2. Power Rule for $n \epsilon \mathbb{Q}$, $\frac{d}{dx}[x^n] = nx^{n-1}$

3. Constant Multiple Rule $\in \mathbb{R}$, $\frac{d}{dx}[cf(x)] = cf'(x)$

4. Sum & Difference Rules $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$

Examples:

$f(x) = 3x^2$ $\frac{d}{dx} f(x) = (3) \cdot \frac{d}{dx}(x^2)$

$f'(x) = 3 \cdot (x^2)' = 3 \cdot (2x) = \boxed{6x}$

$f(x) = \frac{3}{x} = 3x^{-1}$

$f'(x) = 3(x^{-1})' = 3(-1x^{-1-1}) = \boxed{-3x^{-2}}$

$g(x) = 2x^3 - x^2 + 3x$

$g'(x) = 2(3x^{3-1}) - (2x^{2-1}) + 3(1x^{1-1}) = \boxed{6x^2 - 2x + 3}$

$y = 4x^{3/2} - 5x^4 + 2x^{\frac{1}{3}} - 7$

$y' = 4\left(\frac{3}{2}x^{\frac{3}{2}-1}\right) - 5(4x^{4-1}) + 2\left(\frac{1}{3}x^{\frac{1}{3}-1}\right) - 0$

$= \boxed{6x^{1/2} - 20x^3 + \frac{2}{3}x^{-2/3}}$

$6\sqrt{x} - 20x^3 + \frac{2}{3x^{2/3}}$

Derivatives of Trig Functions

1. $\frac{d}{dx}[\sin x] = \cos x$
2. $\frac{d}{dx}[\cos x] = -\sin x$
3. $\frac{d}{dx}[\tan x] = \sec^2 x$
4. $\frac{d}{dx}[\cot x] = -\csc^2 x$
5. $\frac{d}{dx}[\sec x] = \sec x \tan x$
6. $\frac{d}{dx}[\csc x] = -\csc x \cot x$

Proof of $\frac{d}{dx}[\sin x] = \cos x$:

$$\frac{d}{dx}[\sin x] = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} =$$

$$= \lim_{h\to 0} \frac{\sin x \cosh + \cos x \sin h - \sin x}{h} =$$

$$= \lim_{h\to 0} \underbrace{\cos x}_{\cos x} \cdot \underbrace{\frac{\sin h}{h}}_{1} + \lim_{h\to 0} \underbrace{\sin x}_{\sin x} \cdot \underbrace{\frac{(\cos h - 1)}{h}}_{-(0)}$$

$$\cos x \cdot 1 + \sin x \cdot -(0)$$

$$= \boxed{\cos x}$$

Homework:

Find the derivative by the limit process:
2.1 #21, 23

Use the alternate form to find the derivative:
2.1 #61-69 odd

Describe the x-values where the function is differentiable (given a graph):
2.1 #71-79 odd

Find the derivative using the basic derivative rules we have learned so far:
2.2 #3-51 odd

Work through intuitive exercises on **Khan Academy**:
- Slope of secant lines
- Tangent lope is limiting value of secant slope
- Derivative intuition
- Visualizing derivatives
- Graphs of functions and their derivatives
- The formal and alternate form of the derivative
- Derivatives 1
- Recognizing slopes of curves
- Power rule
- Special derivatives