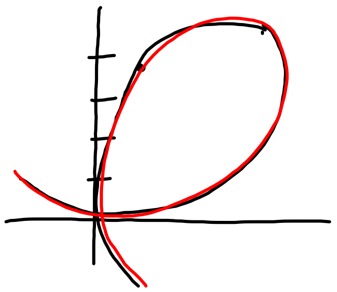
32. Folium of Descartes

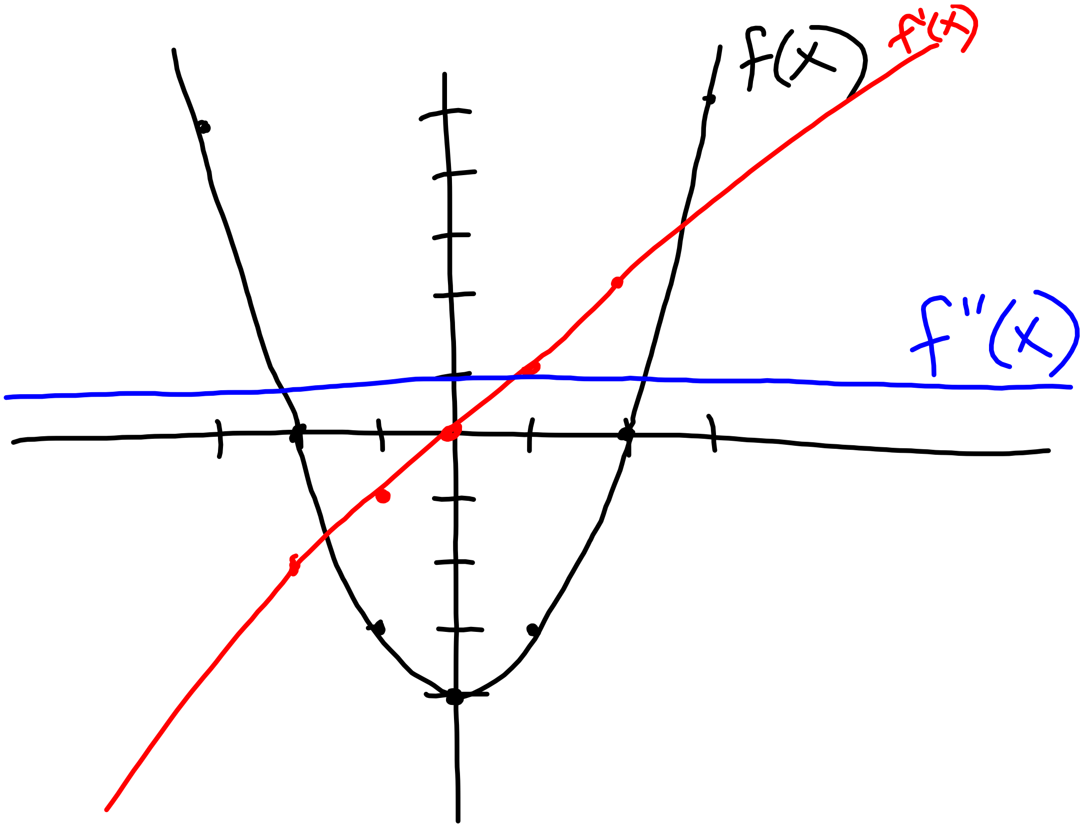


$x^3+y^3-6xy=0$

find the slope of the tangent line @ $\left(\frac{4}{3}, \frac{8}{3}\right)$

$$3x^2+3y^2y'-6xy'-6y=0$$
$$y'(3y^2-6x)=6y-3x^2$$
$$y'=\frac{6y-3x^2}{3y^2-6x}=\frac{2y-x^2}{y^2-2x}$$
$$m=y'\Big|_{\left(\frac{4}{3},\frac{8}{3}\right)}=\frac{2\left(\frac{8}{3}\right)-\left(\frac{4}{3}\right)^2}{\left(\frac{8}{3}\right)^2-2\left(\frac{4}{3}\right)}=\frac{\frac{16}{3}-\frac{16}{9}}{\frac{64}{9}-\frac{8}{3}}=\frac{\frac{32}{9}}{\frac{40}{9}}=\frac{\cancel{32}^{\,8}}{\cancel{9}}\cdot\frac{\cancel{9}}{\cancel{40}_{\,10}}$$

$$= \boxed{\frac{4}{5}}$$



40. Find y'' in terms of x & y.

$y^2 = 4x$

$2yy' = 4$

$y' = \frac{4}{2y} = \frac{2}{y} = 2y^{-1}$

$y'' = -2y^{-2} \cdot y' = (-2y^{-2})(2y^{-1})$

$= -4y^{-3} = \boxed{-\frac{4}{y^3}}$

38. Find y'' in terms of x & y

$1 - xy = x - y$

$(-xy)' = (-x)'y + (-x)y'$

$= (-1)y + (-x)y'$

$= -y - xy'$

$-xy' - y = 1 - y'$

$y' - xy' = 1 + y$

$y'(1-x) = 1 + y$

$y' = \frac{1+y}{1-x}$

$y'' = \frac{(1-x)(1+y)' - (1+y)(1-x)'}{(1-x)^2}$

$y'' = \frac{(1-x)y' - (1+y)(-1)}{(1-x)^2}$

$y'' = \frac{(\cancel{1-x})\frac{(1+y)}{\cancel{1-x}} + 1 + y}{(1-x)^2}$

$\boxed{y'' = \frac{2+2y}{(1-x)^2}}$

Rates of Change

The **average rate of change** of a function $f(x)$ from x_1 to x_2 is the slope of the secant line containing the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}$$

The **instantaneous rate of change** of a function $f(x)$ at a particular value x_0 is the slope of the tangent line at the point $(x_0, f(x_0))$, i.e. $f'(x_0)$.

Example: $y=x^2-2$

(a) Find the average rate of change of y with respect to x over the interval [2,4].

(b) Find the instantaneous rate of change of y with respect to x when $x=4$.

(a) $\frac{\Delta y}{\Delta x}=\frac{(4^2-2)-(2^2-2)}{4-2}=\frac{14-2}{2}=\frac{12}{2}=\boxed{6}$

(b) $y'=2x\Big|_{x=4}=2(4)=\boxed{8}$

Example: A particle moves on a line away from its initial position so that after t seconds it is $s(t)=2t^2-t$ feet from its initial position.

(a) Find the average velocity of the particle over the interval [1,3].

(b) Find the instantaneous velocity at $t=2$.

*Note that velocity is rate of change of position.

(a) $\frac{\Delta s}{\Delta t}=\frac{[2(3)^2-3]-[2(1)^2-1]}{3-1}=\boxed{7\ \text{ft/s}}$

(b) $v(t)=s'(t)=4t-1$

$s'(2)=4(2)-1=\boxed{7\ \text{ft/s}}$

2.6 Related Rates

18. The radius r of a sphere is increasing at a rate of 2 inches per minute.
(a) Find the rate of change of the volume when $r=6$ inches and $r=24$ inches.
(b) Explain why the rate of change of the volume of the sphere is not constant even though dr/dt is constant.

$$V=\frac{4}{3}\pi r^3 \qquad \frac{dr}{dt}=2\ \text{in}/\text{min}$$

$$\frac{dV}{dt}=?\ \text{when } r=6$$

$$\frac{d}{dt}[V]=\frac{d}{dt}\left[\frac{4}{3}\pi r^3\right]$$

$$\frac{dV}{dt}=\frac{4}{3}\pi(3r^2)\cdot\frac{dr}{dt} \Rightarrow \frac{dV}{dt}=8\pi r^2$$

$$\frac{dV}{dt}=4\pi r^2\cdot\frac{dr}{dt}\Bigg|_{\frac{dr}{dt}=2,\ r=6}=4\pi(6)^2\cdot 2=\boxed{288\pi\ \frac{\text{in}^3}{\text{min}}}$$

$$\frac{dV}{dt}=8\pi r^2$$

22. Find the rate of change of the volume of a cone if dr/dt is 2 inches per minute and $h=3r$ when (a) $r=6$ inches and (b) $r=24$ inches.

$$V=\frac{1}{3}\pi r^2 h \qquad \frac{dV}{dt}=?\ \text{when } r=6;\ \frac{dr}{dt}=2\text{in}/\text{min}$$

$$h=3r$$

$$\cancel{\frac{d}{dt}[V]=\frac{d}{dt}\left[\frac{1}{3}\pi r^2 h\right]}$$

$$\cancel{\frac{dV}{dt}=\frac{1}{3}\pi r^2\cdot\frac{dh}{dt}+\frac{1}{3}\pi\left(2r\cdot\frac{dr}{dt}\right)\cdot h}$$

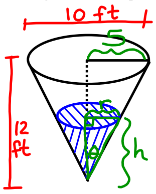
$$V=\frac{1}{3}\pi r^2(3r)$$

$$V=\pi r^3$$

$$\frac{dV}{dt}=3\pi r^2\cdot\frac{dr}{dt}=3\pi r^2(2)=6\pi r^2$$

$$\frac{dV}{dt}\Bigg|_{r=6}=6\pi(6)^2=\boxed{216\pi\ \frac{\text{in}^3}{\text{min}}}$$

24. A conical tank (with vertex down) is 10 feet across the top and 12 feet deep. If water is lowing into the tank at a rate of 10 cubic feet per minute, ind the rate of change of the depth of the water when the water is 8 feet deep.



$\frac{dV}{dt} = 10\,ft^3/min$

$\frac{dh}{dt} = ?$ when $h = 8\,ft$

$\frac{r}{h} = \frac{5}{12}$ $\quad 12r = 5h \quad$ $r = \frac{5}{12}h$

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{5}{12}h\right)^2 \cdot h$$

$$V = \frac{1}{3}\pi \cdot \frac{25}{144}h^3$$

$$\frac{dV}{dt} = \pi \cdot \frac{25}{144}h^2 \cdot \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{\frac{dV}{dt}}{\frac{25\pi h^2}{144}} = \frac{10}{\frac{25\pi \cdot 8^2}{144}}$$

$$= \frac{\cancel{10}^{2} \cdot \cancel{144}^{\cancel{18}\,9}}{\cancel{25}_{5} \cdot \cancel{64}_{\cancel{8}\,\cancel{4}\,2}\,\pi} = \boxed{\frac{9}{10\pi}\ ft/min}$$

2.5 # 1-39 odd; 43, 47

2.6 # 15-23 odd