Find $f'(x)$

$f(x) = 2^{\cos x}\operatorname{arccot}(6x)$

$$f'(x) = \left(2^{\cos x}\right)'\operatorname{arccot}(6x) + 2^{\cos x}\left[\operatorname{arccot}(6x)\right]'$$

$$= \boxed{2^{\cos x}\ln 2\,(-\sin x)\operatorname{arccot}(6x) + 2^{\cos x}\cdot\frac{-1}{1+(6x)^2}\cdot 6}$$

$$f(x) = \frac{(7x^3 - 5x^2 + 4x)^{2/3}}{\ln(6x - 11)}$$

$$f'(x) = \boxed{\frac{\ln(6x-11)\cdot\frac{2}{3}(7x^3-5x^2+4x)^{-1/3}\cdot(21x^2-10x+4)}{(\ln(6x-11))^2} - \frac{(7x^3-5x^2+4x)^{2/3}\cdot\frac{1}{6x-11}\cdot 6}{(\ln(6x-11))^2}}$$

9. Find y' implicitly in terms of x and y.

$x^2y + 3xy^3 = 5x^3y^2$

$$2xy + x^2y' + 3y^3 + 3x(3y^2y') = 15x^2y^2 + 5x^3(2yy')$$

$$x^2y' + 9xy^2y' - 10x^3yy' = 15x^2y^2 - 2xy - 3y^3$$

$$y'(x^2 + 9xy^2 - 10x^3y) = 15x^2y^2 - 2xy - 3y^3$$

$$\boxed{y' = \frac{15x^2y^2 - 2xy - 3y^3}{x^2 + 9xy^2 - 10x^3y}}$$

10. Find y' implicitly in terms of x and y.

$\cos x + \sin y = \tan(xy)$

$$-\sin x + y'\cos y = \sec^2(xy) \cdot (y + xy')$$

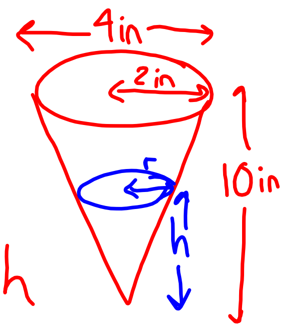
$$-\sin x + y'\cos y = y\sec^2(xy) + xy'\sec^2(xy)$$

$$y'\left(\cos y - x\sec^2(xy)\right) = y\sec^2(xy) + \sin x$$

$$\boxed{y' = \frac{y\sec^2(xy) + \sin x}{\cos y - x\sec^2 xy}}$$

1. A jumbo waffle cone from Sarah's Tasty Ice Cream Shoppe is 10 inches tall and has a 4 inch diameter at the top of the cone. Yesterday, my cone had a leak! Instead of eating it super fast, I decided to compare the rate of change of volume of ice cream to the rate of change of height of ice cream in the cone. How fast is the ice cream leaking out (in cubic inches per minute) when there are 5 inches of ice cream in the cone, if the height of ice cream in the cone is changing at a rate of 1 inch every 5 minutes?

$\frac{dh}{dt} = \frac{1\text{ in}}{5\text{ min}}$; $\frac{dV}{dt} = ?\ \text{in}^3/\text{min}$ when $h = 5\text{in}$



$$\frac{r}{h} = \frac{2}{10} \Rightarrow r = \frac{h}{5}$$

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{5}\right)^2 \cdot h$$

$$V = \frac{\pi}{3 \cdot 25} h^3$$

$$\frac{dV}{dt} = \frac{\pi}{25} h^2 \cdot \frac{dh}{dt}$$

$$= \frac{\pi}{25} \cdot (5\text{in})^2 \cdot \frac{1\text{in}}{5\text{min}} = \boxed{\frac{\pi}{5}\ \text{in}^3/\text{min}}$$

2. $x^3 + y^2 = 10$

a. Find y' in terms of x and y.

b. Find y'' in terms of x and y.

$3x^2 + 2yy' = 0$

$2yy' = -3x^2$

$y' = \frac{-3x^2}{2y}$

$y'' = \frac{2y(-6x) - (-3x^2)(2y')}{(2y)^2}$

$= \boxed{\frac{-12xy + 6x^2\left(\frac{-3x^2}{2y}\right)}{(2y)^2}}$

1. Locate the absolute extrema of the function on the closed interval. $f(x) = x^3 - \frac{3}{2}x^2$, $[-1, 2]$

$f'(x) = 3x^2 - 3x$

$3x(x-1) = 0$

$\underbrace{x = 0, 1}_{\text{critical\#'s}}$

$f(-1) = (-1)^3 - \frac{3}{2}(-1)^2 = -1 - \frac{3}{2}$

$= \frac{-2}{2} - \frac{3}{2} = \frac{-5}{2}$

$f(0) = 0$

$f(1) = 1 - \frac{3}{2} = \frac{2}{2} - \frac{3}{2} = \frac{-1}{2}$

$f(2) = 2^3 - \frac{3}{2}(2)^2 = 8 - 6 = 2$

abs. max: 2
when $x = 2$
abs. min: $-5/2$
when $x = -1$

2. Determine if Rolle's Theorem can be applied to f on the closed interval [a , b]. If Rolle's Theorem can be applied, find all values of c in the open interval (a, b) such that f'(c)=0.

$$f(x) = (x-3)(x+1)^2\ , \qquad [-1,3]$$

f is cts on $[-1,3]$
f is diff on $(-1,3)$
$f(-1)=(-1-3)(-1+1)^2=0$
$f(3)=(3-3)(3+1)^2=0$
$f(-1)=f(3)$
Rolle's Theorem applies

$f(x)=(x-3)(x^2+2x+1)$
$=x^3+2x^2+x-3x^2-6x-3$
$=x^3-x^2-5x-3$
$f'(x)=3x^2-2x-5$

$3x^2-5x+3x-5=0$
$x(3x-5)+1(3x-5)=0$
$(3x-5)(x+1)=0$
$\boxed{x=5/3}$, $\cancel{x=-1}$ not in open interval

3. Determine whether the Mean Value Theorem can be applied to f on the closed interval [a, b]. If the Mean Value Theorem can be applied, find all values of c in the open interval (a, b) such that $f'(c)=\frac{f(b)-f(a)}{b-a}$. $f(x) = x(x^2-x-2)$, $[-1,1]$

f is cts on $[-1,1]$
f is diff on $(-1,1)$
$\Rightarrow$ MVT applies

$\dfrac{f(b)-f(a)}{b-a} = \dfrac{1(1-1-2)-(-1)(1+1-2)}{1-(-1)} = \dfrac{-2}{2} = -1$

$f(x)=x^3-x^2-2x$
$f'(x)=3x^2-2x-2$

$3x^2-2x-2=-1$
$3x^2-2x-1=0$
$3x^2-3x+x-1=0$
$3x(x-1)+1(x-1)=0$
$(x-1)(3x+1)=0$
$\cancel{x=1}$ $\boxed{x=-1/3}$

4. Find the open intervals on which the function is increasing or decreasing and locate all relative extrema. $f(x) = (x+2)^2(x-1)$

$f(x) = (x^2+4x+4)(x-1)$

$= x^3+4x^2+4x-x^2-4x-4$

$= x^3+3x^2-4$

$f'(x) = 3x^2+6x$

$3x(x+2) = 0$

$\underbrace{x = 0, -2}_{\text{critical \#'s}}$

$f(0) = -4$

$f(-2) = 0$

$f'(-3)$ $f'(-1)$ $f'(1)$

-2 0

$+$ ↗ $-$ ↘ $+$ ↗

$$\boxed{\begin{array}{l} f \text{ is increasing on } (-\infty,-2) \cup (0,\infty) \\ f \text{ is decreasing on } (-2,0) \\ \text{rel. max: } 0 \text{ when } x=-2 \\ \text{rel. min: } -4 \text{ when } x=0 \end{array}}$$

5. Find the open intervals on which the function is increasing or decreasing and locate all relative extrema. $y = \frac{x^2}{x^2-9}$

$$y' = \frac{(x^2-9)(2x) - x^2(2x)}{(x^2-9)^2}$$

$$= \frac{2x^3-18x-2x^3}{(x^2-9)^2}$$

$$= \frac{-18x}{(x^2-9)^2}$$

critical #'s: $-3, 0, 3$

f is undefined @ -3 & 3

$f(0) = 0$

$f'(-4)$ $f'(-1)$ $f'(1)$ $f'(4)$

-3 0 3

$+$ ↗ $+$ ↗ $-$ ↘ $-$ ↘

$$\boxed{\begin{array}{l} f \text{ is increasing on } (-\infty,-3) \cup (-3,0) \\ f \text{ is decreasing on } (0,3) \cup (3,\infty) \\ \text{rel. max: } 0 \text{ when } x=0 \end{array}}$$

6. Find the points of inflection and discuss concavity of the graph of the function. $f(x) = x^3(x-4)$

$f(x) = x^4 - 4x^3$

$f'(x) = 4x^3 - 12x^2$

$f''(x) = 12x^2 - 24x$

$12x(x-2) = 0$

$x = 0, 2$

$f''(-1)$ $f''(1)$ $f''(3)$

$+$ 0 $-$ 2 $+$

$\boxed{\begin{array}{l} f \text{ is concave up on } (-\infty, 0) \cup (2, \infty) \\ f \text{ is concave down on } (0, 2) \\ f \text{ has inflection points @ } (0,0) \text{ \& } (2,-16) \end{array}}$

7. Find the points of inflection and discuss concavity of the graph of the function. $f(x) = \frac{x}{x^2+1}$

$$f'(x) = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \frac{x^2+1-2x^2}{(x^2+1)^2}$$

$$= \frac{1-x^2}{(x^2+1)^2}$$

$$f''(x) = \frac{(x^2+1)^2(-2x) - (1-x^2)[2(x^2+1)(2x)]}{(x^2+1)^2}$$

$$= \frac{(x^2+1)[-2x(x^2+1) - 4x(1-x^2)]}{(x^2+1)^2}$$

$$= \frac{-2x^3-2x-4x+4x^3}{x^2+1} = \frac{2x^3-6x}{x^2+1}$$

$$\frac{2x(x^2-3)}{x^2+1} = 0 \quad @\ x = 0, \pm\sqrt{3}$$

$f(-\sqrt{3}) = \frac{-\sqrt{3}}{(-\sqrt{3})^2+1} = \frac{-\sqrt{3}}{4}$

$f''(-2)$ $f''(-1)$ $f''(1)$ $f''(2)$

$-$ $-\sqrt{3}$ $+$ 0 $-$ $\sqrt{3}$ $+$

$f(0) = \frac{0}{0^2+1} = 0$

$f(\sqrt{3}) = \frac{\sqrt{3}}{(\sqrt{3})^2+1} = \frac{\sqrt{3}}{4}$

$\boxed{\begin{array}{l} f \text{ is concave down on } (-\infty, -\sqrt{3}) \cup (0, \sqrt{3}) \\ f \text{ is concave up on } (-\sqrt{3}, 0) \cup (\sqrt{3}, \infty) \\ \text{inflection points: } \left(-\sqrt{3}, \frac{-\sqrt{3}}{4}\right), (0,0), \left(\sqrt{3}, \frac{\sqrt{3}}{4}\right) \end{array}}$

8. Use the Second Derivative Test to find all relative extrema. $f(x) = x^2 - 6x + 7$

$f'(x) = 2x - 6$

have extremum when $x = 3$

$f''(x) = 2 > 0$

$\Rightarrow$ f is concave up

$\Rightarrow (3, -2)$ is a relative minimum

$(c, f(c))$

12. The radius of a right circular cylinder is given by $\sqrt{t+2}$ and its height is $\frac{1}{2}t$, where t is time in seconds and the dimensions are in inches. Find the rate of change of the volume with respect to time. Volume of a cylinder is given by $V = \pi r^2 h$, where r is the radius of the cylinder and h is the height.

$r = \sqrt{t+2}$ $\quad \frac{dV}{dt} = ?$

$h = \frac{1}{2}t$

$V = \pi r^2 h$

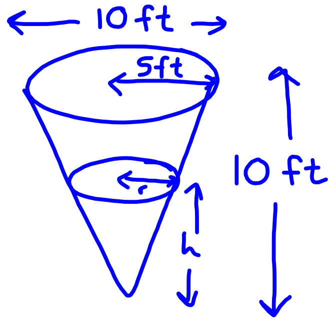
$= \pi\left(\sqrt{t+2}\right)^2\left(\frac{1}{2}t\right)$

$= \pi(t+2)\left(\frac{1}{2}t\right)$

$V = \frac{\pi}{2}t^2 + \pi t$

$\boxed{\frac{dV}{dt} = \pi t + \pi \text{ in}^3/\text{s}}$

13. A conical tank is 10 feet across at the top and 10 feet deep. If it is being filled with water at a rate of 5 cubic feet per minute, find the rate of change of the depth of the water when it is 3 feet deep. The volume of a cone is given by $=\frac{1}{3}\pi r^2 h$, where r is the radius of the cone and h is the height. Give an exact answer in terms of π.



$$V=\frac{1}{3}\pi r^2 h \qquad \frac{dV}{dt}=5\ ft^3/min$$

$$\frac{r}{h}=\frac{5}{10} \qquad \frac{dh}{dt}=?\ \text{when } h=3ft$$

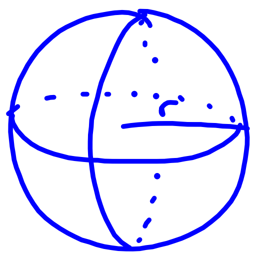
$$r=\frac{h}{2}$$

$$V=\frac{1}{3}\pi\left(\frac{h}{2}\right)^2\cdot h \qquad \frac{dV}{dt}=\frac{\pi}{4}h^2\cdot\frac{dh}{dt}$$

$$=\frac{\pi}{3}\cdot\frac{h^3}{4} \qquad 5\frac{ft^3}{min}=\frac{\pi}{4}(3ft)^2\cdot\frac{dh}{dt}$$

$$\boxed{\frac{20}{9\pi}\ ft/min}=\frac{5\ ft^3/min}{\frac{9\pi}{4}ft^2}=\frac{dh}{dt}$$

14. The radius of a sphere is expanding at a rate of 3 centimeters per second. Find the rate of change of the volume of the ~~cube~~ sphere when the radius is 12 centimeters.



$$V=\frac{4}{3}\pi r^3 \qquad \frac{dr}{dt}=3\ cm/s$$

$$\frac{dV}{dt}=?\ \text{when } r=12cm$$

$$\frac{dV}{dt}=4\pi r^2\cdot\frac{dr}{dt}=4\pi(12cm)^2\cdot 3\ cm/s=\boxed{1728\pi\ \frac{cm^3}{s}}$$

Homework since Test #2 (Material for Test #3)

2.5 # 1-39 odd; 43, 47 - Implicit Differentiation
2.6 # 15-23 odd - Related Rates
2.6 # 25, 27, 35 - Related Rates (more challenging problems)

3.1 # 17-31 odd - Absolute Extrema on an Interval
3.2 # 7-19 odd - Rolle's Theorem
3.2 # 31-37 odd - Mean Value Theorem
3.3 # 11-31 odd - Increasing, Decreasing, and Relative Extrema
3.4 #11-25 odd - Inlection Points and Concavity

position: $s(t)$

velocity: $v(t) = s'(t)$

acceleration: $a(t) = v'(t) = s''(t)$

$$y = 2 + \sqrt{x^2+y^2}$$

$$y = 2 + (x^2+y^2)^{1/2}$$

$$y' = \frac{1}{2}(x^2+y^2)^{-1/2} \cdot \left[2x + 2yy'\right]$$

$$y' = \frac{x}{\sqrt{x^2+y^2}} + \frac{yy'}{\sqrt{x^2+y^2}}$$

$$y' - \frac{yy'}{\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}$$

$$y'\left(1 - \frac{y}{\sqrt{x^2+y^2}}\right) = \frac{x}{\sqrt{x^2+y^2}}$$

$$\boxed{y' = \frac{\frac{x}{\sqrt{x^2+y^2}}}{1 - \frac{y}{\sqrt{x^2+y^2}}}}$$