3.5 Limits at Infinity

$\lim_{x\to\infty} f(x)$ (end behavior)

correspond exactly with horizontal & oblique asymptotes

$$f(x) = \frac{5x^2-3x+4}{2x^2+5x} \approx \frac{5x^2}{2x^2} = \frac{5}{2}$$

Horizontal asymptote @ $y=\frac{5}{2}$

$\Rightarrow \lim_{x\to\infty} f(x) = \frac{5}{2}$ & $\lim_{x\to-\infty} f(x) = \frac{5}{2}$

$$f(x) = \frac{2x-4}{3x^4} \approx \frac{2x}{3x^4} = \frac{2}{3x^3} \to 0$$

Horizontal asymptote @ $y=0$

$\lim_{x\to\pm\infty} f(x) = 0$

$$f(x) = \frac{2x^7 - 4x^3 - 2}{5x^4 + 1} \approx \frac{2x^7}{5x^4} \approx \frac{2}{5}x^3$$

$$\lim_{x\to\infty} f(x) = \infty$$

$$\lim_{x\to-\infty} f(x) = -\infty$$

$$f(x) = \frac{2 - 7x^3 + 2x}{1 + x} \approx \frac{-7x^3}{x} = -7x^2$$

$$\lim_{x\to\infty} f(x) = -\infty$$

$$\lim_{x\to-\infty} f(x) = -\infty$$

24. $$\lim_{x\to-\infty}\left(\frac{1}{2}x - \frac{4}{x^2}\right)$$

$$= \lim_{x\to-\infty} \frac{1}{2}x - \lim_{x\to-\infty} \frac{4}{x^2}$$

$$= -\infty - 0$$

$$= \boxed{-\infty}$$

26. $\lim_{x\to-\infty} \dfrac{x}{\sqrt{x^2+1}} \approx \dfrac{x}{\sqrt{x^2}} \approx \dfrac{x}{|x|}$

$\sqrt[n]{x^n} = \begin{cases} x, & \text{if } n \text{ is odd} \\ |x|, & \text{if } n \text{ is even} \end{cases}$

$\sqrt[3]{(-1)^3} = \sqrt[3]{-1} = -1$

$\sqrt[2]{(-1)^2} = \sqrt[2]{1} = 1 = |-1|$

$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

$\lim_{x\to-\infty} \dfrac{x}{|x|} = \lim_{x\to-\infty} \dfrac{x}{-x} = \lim_{x\to-\infty} (-1)$

$= \boxed{-1}$

28. $\lim_{x\to-\infty} \dfrac{-3x+1}{\sqrt{x^2+1}}$

$= \lim_{x\to-\infty} \dfrac{-3x}{\sqrt{x^2}} = \lim_{x\to-\infty} \dfrac{-3x}{|x|} = \lim_{x\to-\infty} \dfrac{-3x}{-x}$

$= \lim_{x\to-\infty} 3 = \boxed{3}$

$|-2| = 2 = -(-2)$

30. $$\lim_{x\to\infty} \frac{x-\cos x}{x}$$

$$= \underbrace{\lim_{x\to\infty} \frac{x}{x}}_{1} - \underbrace{\lim_{x\to\infty} \frac{\cos x}{x}}_{0} = \boxed{1}$$

$$-\frac{1}{x} \le \frac{\cos x}{x} \le \frac{1}{x}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$0 \qquad 0 \text{ by Squeeze Thm} \qquad 0$$

32. $$\lim_{x\to\infty} \cos\frac{1}{x}$$

$$= \cos\left[\lim_{x\to\infty} \frac{1}{x}\right] = \cos 0 = \boxed{1}$$

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

18. c.

$$\lim_{x\to\infty} \frac{5x^{3/2}}{4\sqrt{x}+1} = \lim_{x\to\infty} \frac{5x^{3/2}}{4x^{1/2}}$$

$$= \lim_{x\to\infty} \frac{5}{4}x^{1} = \boxed{\infty}$$

7.7 Indeterminate Forms & L'Hôpital's Rule

Indeterminate Forms:

$$\underbrace{\frac{0}{0}, \frac{\infty}{\infty}, \frac{-\infty}{-\infty}, -\frac{\infty}{\infty}}, \; 0\cdot\infty,$$

$$1^{\infty}, 0^{0}, \infty-\infty$$

Let f & g be differentiable functions on an open interval containing c, except maybe at c itself. $g'(x) \neq 0$, except maybe at c.

If $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\infty}{\infty}$,

then $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$.

7.7

12. $\lim_{x \to -1} \frac{x^2-x-2}{x+1} = \frac{0}{0}$ L'H applies

$= \lim_{x \to -1} \frac{(x^2-x-2)'}{(x+1)'} = \lim_{x \to -1} \frac{2x-1}{1} = \boxed{-3}$

16. $$\lim_{x\to 0^+} \frac{e^x-(1+x)}{x^3} = \frac{0}{0}$$

$$= \lim_{x\to 0^+} \frac{(e^x-1-x)'}{(x^3)'} = \lim_{x\to 0^+} \frac{e^x-1}{3x^2} = \frac{0}{0}$$

$$= \lim_{x\to 0^+} \frac{(e^x-1)'}{(3x^2)'} = \lim_{x\to 0^+} \frac{e^x}{6x} = \boxed{+\infty}$$

18. $$\lim_{x\to 1} \frac{\ln x^2}{x^2-1} = \frac{0}{0}$$

$$= \lim_{x\to 1} \frac{\frac{1}{x^2}\cdot 2\cancel{x}}{2\cancel{x}} = \boxed{1}$$

20. $\lim_{x\to 0} \frac{\sin ax}{\sin bx} = \frac{0}{0}$

Recall: $\lim_{x\to 0} \frac{\sin x}{x} = 1$

$= \lim_{x\to 0} \frac{a \cdot \cos ax}{b \cdot \cos bx} = \boxed{\frac{a}{b}}$

1

1

28. $\lim_{x\to\infty} \frac{x^2}{e^x} = \frac{\infty}{\infty}$

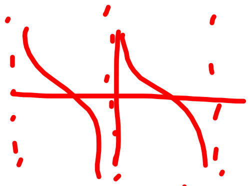
$= \lim_{x\to\infty} \frac{2x}{e^x} = \frac{\infty}{\infty}$

$= \lim_{x\to\infty} \frac{2}{e^x} = \boxed{0}$

36. $\lim\limits_{x\to\infty} \dfrac{e^{x/2}}{x} = \dfrac{\infty}{\infty}$

$= \lim\limits_{x\to\infty} \dfrac{\frac{1}{2}e^{x/2}}{1} = \boxed{\infty}$

38. $\lim\limits_{x\to 0^+} x^3 \cot x = 0\cdot\infty$



cot x

$= \lim\limits_{x\to 0^+} \dfrac{x^3}{\tan x} = \dfrac{0}{0}$

$= \lim\limits_{x\to 0^+} \dfrac{3x^2}{\sec^2 x} = \dfrac{0}{1} = \boxed{0}$

40. $\lim_{x\to\infty} x\tan\frac{1}{x} \approx \infty\cdot 0$

$= \lim_{x\to\infty} \frac{\tan\frac{1}{x}}{\frac{1}{x}} = \frac{0}{0}$

$(x^{-1})' = -x^{-2}$

$= \lim_{x\to\infty} \frac{\sec^2\frac{1}{x}\cdot \cancel{\frac{-1}{x^2}}}{\cancel{\frac{-1}{x^2}}}$

$= \lim_{x\to\infty} \sec^2\frac{1}{x} = \boxed{1}$

42. $\lim_{x\to 0^+} (e^x + x)^{2/x} \approx 1^\infty$

$y = \lim_{x\to 0^+} (e^x + x)^{2/x}$

$\ln y = \ln\left(\lim_{x\to 0^+} (e^x + x)^{2/x}\right)$

$\ln y = \lim_{x\to 0^+} \left[\ln(e^x + x)^{2/x}\right]$

$\ln y = \lim_{x\to 0^+} \frac{2}{x}\ln(e^x + x)$

$\ln y = \underbrace{\lim_{x\to 0^+} \frac{2\ln(e^x + x)}{x}}_{\text{L'H applies } \frac{0}{0}}$

$\ln y = \lim_{x\to 0^+} \frac{\frac{2}{e^x + x}\cdot(e^x + 1)}{1}$

$\ln y = \frac{2\cdot(1+1)}{1+0}$

$\ln y = 4$

$e^{\ln y} = e^4$

$\boxed{y = e^4}$

7.7 HW 11-35
odd