2.1 The Derivative & The Tangent Line Problem

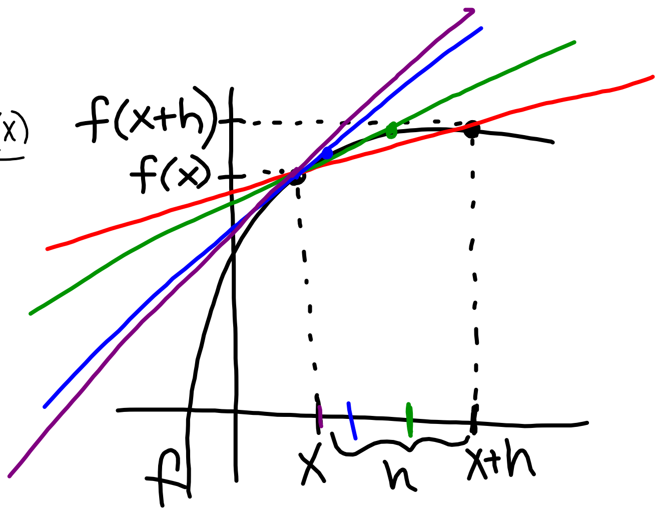
secant line
crosses through a function at two points

slope of the secant line:

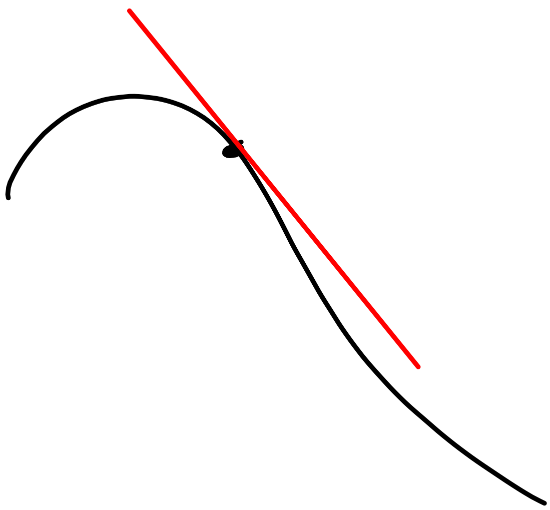
$$\frac{f(x+h)-f(x)}{x+h-x}=\frac{f(x+h)-f(x)}{h}$$

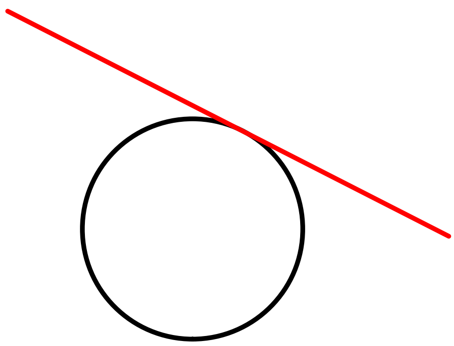
what happens as $h \to 0$?

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$



As $h \to 0$, the secant line approximates the tangent line, and the limit is the slope of the tangent line and we call it **the derivative of f at x**.





$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$\boldsymbol{f'(x)}$ "f prime of x"

$\boldsymbol{\frac{dy}{dx}}$ "derivative of y with respect to x"

$\boldsymbol{y'}$ "y prime"

$\boldsymbol{\frac{d}{dx}[f(x)]}$ "the derivative with respect to x of $f(x)$"

$\boldsymbol{D_x[y]}$ "the partial derivative with respect to x of y"

The Derivative

The slope of the tangent line to the graph of f at the point $(c, f(c))$ is given by:

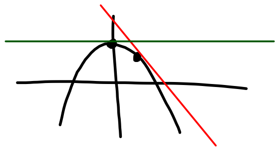
$$m = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x}$$

The derivative of f at x is given by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

8. $g(x) = 5 - x^2$

find slope of tangent line at the points $(2,1)$ & $(0,5)$



slope of $g(x)$ @ $(c, g(c))$ is

$$g'(c) = \lim_{h\to 0} \frac{g(c+h)-g(c)}{h}$$

$(c, g(c)) = (2,1)$:

$$m = g'(2) = \lim_{h\to 0} \frac{g(2+h)-g(2)}{h} = \lim_{h\to 0} \frac{5-(2+h)^2-1}{h}$$

$$= \lim_{h\to 0} \frac{5-(4+4h+h^2)-1}{h} = \lim_{h\to 0} \frac{\cancel{4}-\cancel{4}-4h-h^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(-4-h)}{\cancel{h}} = -4-0 = \boxed{-4}$$

$(c, g(c)) = (0,5)$:

$$m = g'(0) = \lim_{h\to 0} \frac{g(0+h)-g(0)}{h} = \lim_{h\to 0} \frac{\cancel{5}-h^2-\cancel{5}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(-h)}{\cancel{h}} = -0 = \boxed{0}$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = -2x$$

20. $f(x) = x^3 + x^2$

find the derivative

$(a+b)^0$ 1

$(a+b)^1$ 1 1

$(a+b)^2$ 1 2 1

$(a+b)^3$ 1 3 3 1

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} =$$

$$= \lim_{h\to 0} \frac{(x+h)^3+(x+h)^2-(x^3+x^2)}{h} =$$

$$= \lim_{h\to 0} \frac{\cancel{x^3}+3x^2h+3xh^2+h^3+\cancel{x^2}+2xh+h^2-\cancel{x^3}-\cancel{x^2}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(3x^2+3xh+h^2+2x+h)}{\cancel{h}} = \boxed{3x^2+2x}$$

Find the equation of the tangent line to $f(x) = x^3 - x$ at the point $(2, 6)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3 - (x+h) - (x^3 - x)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x} - h - \cancel{x^3} + \cancel{x}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2 - 1)}{\cancel{h}} = 3x^2 - 1$$

$$m = f'(2) = 3(2)^2 - 1 = 11$$

$$y - y_1 = m(x - x_1)$$

$$y - 6 = 11(x-2)$$

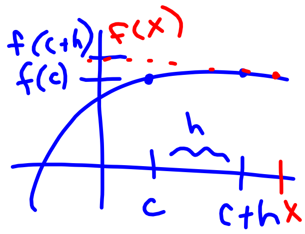
$$y - 6 = 11x - 22$$

$$\boxed{y = 11x - 16}$$

2.1 Differentiability & Continuity

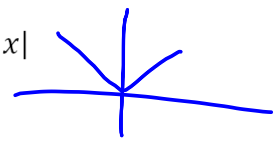
Alternative definition of the derivative at the point $(c, f(c))$:

$$f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$$



All differentiable functions are continuous, but not all continuous functions are differentiable.

e.g. $f(x) = |x|$



$$\lim_{x\to 0^-} \frac{|x| - |0|}{x - 0} = \lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = -1$$

$$\lim_{x\to 0^+} \frac{|x| - |0|}{x - 0} = \lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x} = 1$$

Because left- & right-hand limits are different, the limit in general, and hence the derivative, does not exist

$$f(x) = \sqrt{x}$$

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

$$\lim_{x \to 0^+} \frac{\sqrt{x} - \sqrt{0}}{x - 0}$$

$$\frac{\cancel{x}^m}{\cancel{x}^n} = \frac{x^{m-n}}{1} = \frac{1}{x^{n-m}}$$

$$= \lim_{x \to 0^+} \frac{\sqrt{x}}{x} = \lim_{x \to 0^+} \frac{x^{1/2}}{x^1} = \lim_{x \to 0^+} \frac{1}{x^{1/2}}$$

$$= \lim_{x \to 0^+} \frac{1}{\sqrt{x}} = \infty$$

vertical tangent line @ (0,0)

$\Rightarrow$ Derivative does not exist

2.2 Basic Differentiation Rules

1. The derivative of a constant function is zero, i.e.,

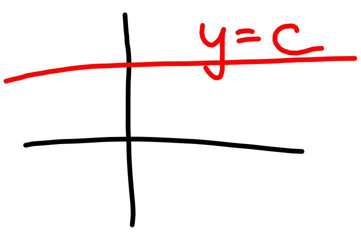
for $c \in \mathbb{R}$, $\frac{d}{dx}[c] = 0$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Proof:

$$[c]' = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} \frac{0}{h} =$$

$$= \lim_{h \to 0} 0 = \boxed{0}$$



2. Power Rule for $n \epsilon \mathbb{Q}$, $\frac{d}{dx}[x^n] = nx^{n-1}$

Proof:

Recall the binomial expansion:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \cdots + \frac{n!}{k!\,(n-k)!}a^{n-k}b^k + \cdots + b^n$$

Special case: $\frac{d}{dx}[x] = 1$

$x = x^1$

$(x)' = 1 \cdot x^{1-1} = 1 \cdot x^0 = 1$

Homework for Test #2 on Derivatives

HW #4 (due Fri 12/05)

- 2.1 #1-23 odd — Find the derivative by the limit process
  - #29-32 all — find the equation of the tangent line
  - #61-69 odd — Use the alternate form to find the derivative
  - #71-79 odd — Describe x-values where the function is differentiable (given graph)
- 2.2 #3-51 odd — Find the derivative using the basic derivative rules
  - #91-94 all; 101, 102 — use the derivative to solve rate of change word problems
- 2.3 #1-53 odd, 63-69 odd, 75-81 all, 83-91 odd, 109-115 all — Product and quotient rules
- 2.4 #7-33 odd — Chain rule

HW #5

- 2.4 #47-81 odd — Chain rule
- 5.1 #45-61, 71 — Logarithmic functions
- 5.4 #39-57 — Exponential functions
- 5.5 #41-55 — Log and exp functions with other bases
- 5.8 #41-59 — Inverse trig functions