54. $$\lim_{x\to 0} \frac{(\sqrt{2+x}-\sqrt{2})}{x} \cdot \frac{(\sqrt{2+x}+\sqrt{2})}{(\sqrt{2+x}+\sqrt{2})}$$

$$(a-b)(a+b) = a^2 - b^2$$

$$= \lim_{x\to 0} \frac{2+x-2}{x(\sqrt{2+x}+\sqrt{2})} = \lim_{x\to 0} \frac{\cancel{x}\ 1}{\cancel{x}(\sqrt{2+x}+\sqrt{2})}$$

$$= \lim_{x\to 0} \frac{1}{\sqrt{2+x}+\sqrt{2}} = \frac{1}{\sqrt{2}+\sqrt{2}} = \boxed{\frac{1}{2\sqrt{2}}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \boxed{\frac{\sqrt{2}}{4}}$$

58. $$\lim_{x\to 0} \frac{\frac{1}{x+4} - \frac{1}{4}}{x} = \lim_{x\to 0} \frac{\frac{1}{x+4}\cdot\frac{4}{4} - \frac{1}{4}\cdot\frac{x+4}{x+4}}{\frac{x}{1}} =$$

$$= \lim_{x\to 0} \frac{4-(x+4)}{4(x+4)} \cdot \frac{1}{x} = \lim_{x\to 0} \frac{-\cancel{x}\ 1}{4(x+4)\cdot\cancel{x}}$$

$$= \lim_{x\to 0} \frac{-1}{4(x+4)} = \boxed{\frac{-1}{16}}$$

66. $\lim_{x\to 2}\frac{x^5-32}{x-2} = \lim_{x\to 2}\frac{\cancel{(x-2)}(x^4+2x^3+4x^2+8x+16)}{\cancel{x-2}}$

$$\begin{array}{r|rrrrrr} 2 & 1 & 0 & 0 & 0 & 0 & -32 \\ & \downarrow & 2 & 4 & 8 & 16 & 32 \\ \hline & 1 & 2 & 4 & 8 & 16 & \boxed{0} \end{array}$$

$= \lim_{x\to 2}(x^4+2x^3+4x^2+8x+16)$

$= 5\cdot 2^4 = \boxed{80}$

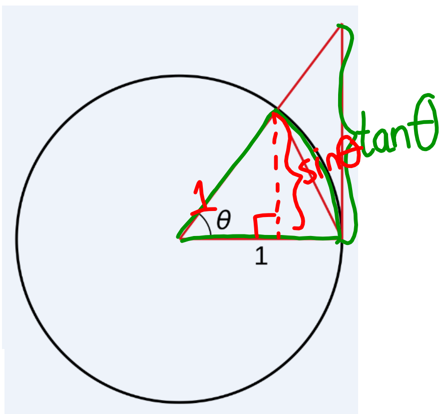
1.3 The Squeeze Theorem

$\lim_{x\to 0}\frac{\sin x}{x} = ?$

Area of whole circle $= \pi r^2|_{r=1} = \pi$

$\frac{\text{Area of whole circle}}{\text{Total angle of circle}} = \frac{\text{Area of sector}}{\theta}$

$\frac{\pi}{2\pi} = \frac{\text{Area of sector}}{\theta} \rightarrow$ Area of sector $= \frac{\theta}{2}$



Area of outer triangle $\geq$ Area of sector $\geq$ Area of inner triangle

$$\frac{\tan\theta}{2} \geq \frac{\theta}{2} \geq \frac{\sin\theta}{2}$$

Multiply through by $\frac{2}{\sin\theta}$

$$\frac{\sin\theta}{2\cos\theta}\cdot\frac{2}{\sin\theta} \geq \frac{\theta}{2}\cdot\frac{2}{\sin\theta} \geq \frac{\sin\theta}{2}\cdot\frac{2}{\sin\theta}$$

$$\frac{1}{\cos\theta} \geq \frac{\theta}{\sin\theta} \geq 1$$

Take reciprocals and reverse inequalities

$$\cos\theta \leq \frac{\sin\theta}{\theta} \leq 1$$

Take limits

$$\lim_{\theta \to 0} \cos\theta \quad \le \quad \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \quad \le \quad \lim_{\theta \to 0} 1$$

$$1 \le \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \le 1$$

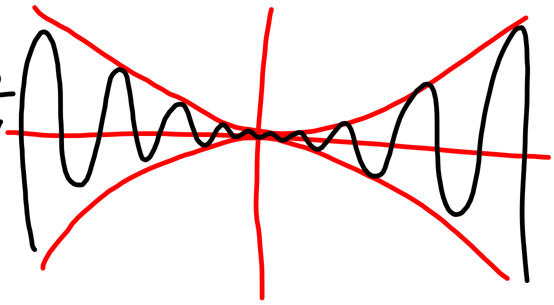
$$\Rightarrow \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = \boxed{1}$$

<u>The Squeeze Theorem</u>:

If $f(x) \le g(x) \le h(x)$ *and* $\lim_{x \to c} f(x) = L = \lim_{x \to c} h(x)$,

Then $\lim_{x \to c} g(x) = L$.

$$-x^2 \le x^2 \sin x \le x^2$$



Special Limits Derived by Squeeze Theorem:

$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1} \ ; \ \boxed{\lim_{x \to 0} \frac{1 - \cos x}{x} = 0}$$

Memorize!!

Use the squeeze theorem to find

$$\lim_{x\to 0}\left(x^2\cos\frac{5}{x} - 3\right)$$

$-1 \le \cos\theta \le 1$

$$-1 \le \cos\frac{5}{x} \le 1$$

$$-x^2 \le x^2\cos\frac{5}{x} \le x^2$$

$$-x^2-3 \le x^2\cos\frac{5}{x} - 3 \le x^2 - 3$$

$$\lim_{x\to 0}(-x^2-3) \le \lim_{x\to 0}\left(x^2\cos\frac{5}{x}-3\right) \le \lim_{x\to 0}(x^2-3)$$

$$-3 \le \lim_{x\to 0}\left(x^2\cos\frac{5}{x}-3\right) \le -3$$

By the Squeeze Theorem,

$$\lim_{x\to 0}\left(x^2\cos\frac{5}{x}-3\right) = \boxed{-3}$$

68. $\displaystyle\lim_{x\to 0}\frac{3(1-\cos x)}{x}$

$\displaystyle\lim_{x\to 0}\frac{1-\cos x}{x} = 0$

$$= \lim_{x\to 0}\left(\frac{3}{1}\right)\left(\frac{1-\cos x}{x}\right)$$

$$= 3\cdot\lim_{x\to 0}\left(\frac{1-\cos x}{x}\right)$$

$$= 3\cdot 0$$

$$= \boxed{0}$$

72. $\lim_{x\to 0} \frac{\tan^2 x}{x}$ $= \lim_{x\to 0} \frac{\sin^2 x}{x\cos^2 x} =$

$= \lim_{x\to 0} \left(\frac{\sin x}{x}\right)\left(\frac{\sin x}{\cos^2 x}\right) =$

$= \left(\lim_{x\to 0} \frac{\sin x}{x}\right)\left(\lim_{x\to 0} \frac{\sin x}{\cos^2 x}\right) =$

$= 1 \cdot 0$

$= \boxed{0}$

78. $\lim_{x\to 0} \frac{\sin 2x}{\sin 3x}$

$\lim_{2x\to 0} \frac{\sin 2x}{2x} = 1$

$\Leftrightarrow \lim_{x\to 0} \frac{\sin 2x}{2x} = 1$

$= \lim_{x\to 0} \frac{\sin 2x}{2x} \cdot \frac{3x}{\sin 3x} \cdot \frac{2}{3}$

$= \frac{2}{3} \lim_{x\to 0} \frac{\frac{\sin 2x}{2x}}{\frac{\sin 3x}{3x}} = \frac{\frac{2}{3}\left(\lim_{x\to 0} \frac{\sin 2x}{2x}\right)}{\left(\lim_{x\to 0} \frac{\sin 3x}{3x}\right)}$

$= \frac{\frac{2}{3}\cdot 1}{1} = \boxed{\frac{2}{3}}$

Homework #2 - due next Fri, 8/22:

1.3 #11,17,27-35odd, 39-61odd

1.3 #67-77odd; 87, 88

Quiz #1 - Wed, 8/20