3.9 - Differentials

Recall:
For a function f that is differentiable at c, the equation of the tangent line at the point $(c, f(c))$ is given by

$y - f(c) = f'(c)(x - c)$

This follows from the point-slope equation $y - y_1 = m(x - x_1)$, where the slope m is the derivative $f'(x)$ evaluated at the point $(c, f(c))$.

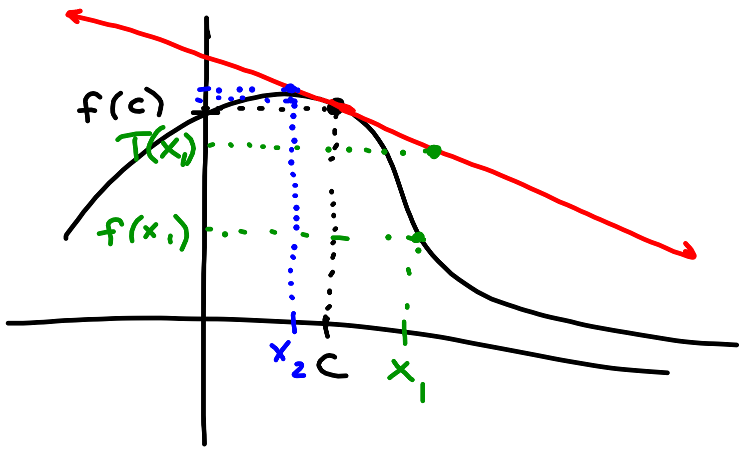
Since c, $f(c)$, and $f'(c)$ are all constants, if we rearrange to solve for y,

$y = f(c) + f'(c)(x - c)$

y is a linear function of x, called the linear approximation or tangent line approximation to the graph of $f(x)$ at $x = c$.

$T(x) = f(c) + f'(c)(x - c)$

For values of x close to c, values of $y = T(x)$ can be used as approximations of the values of the original function f.



Recall that the slope of the *secant line* through two points $(c, f(c))$ and $(x, f(x))$ is given by $\frac{\Delta y}{\Delta x} = \frac{f(x)-f(c)}{x-c}$, and the slope of the *tangent line* is the limit as the distance between these two points goes to zero of this expression, which we define to be the derivative.

Noting that the change in x is $\Delta x = x - c$, or $x = c + \Delta x$ and hence $f(x) = f(c + \Delta x)$, we can write this two ways:

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x}$$

Actual change in y is $\Delta y = f(x) - f(c) = f(c + \Delta x) - f(c)$.

Recalling the tangent line *approximation* equation

$T(x) = f(c) + f'(c)(x - c) = f(c) + f'(c)\Delta x$

We can see that change in y can be approximated by $T(x) - f(c)$, or

Approximate change in y is $\Delta y \approx f'(c)\Delta x$.

For such an approximation, Δx is denoted dx, and is called the differential of x. The expression $f'(x)dx$ is denoted by dy and called the **differential of y**.

$$\boldsymbol{dy = f'(x)dx}$$

In many applications, the differential of y can be used as an approximation of the actual change in y, i.e. $\Delta y \approx f'(x)dx$

3.9 #2 $f(x) = \frac{6}{x^2}$; $\left(2, \frac{3}{2}\right)$

Compare the actual function values with the tangent line approximation near 2.

$f(x) = 6x^{-2}$ $f(2) = \frac{3}{2}$

$f'(x) = -12x^{-3} = -\frac{12}{x^3}$

$f'(2) = \frac{-12}{2^3} = \frac{-12}{8} = \frac{-3}{2}$ $c = 2$

Tangent line $T(x)$: $y = f(c) + f'(c)(x - c)$

$T(x) = f(2) + f'(2)(x-2)$

$T(x) = \frac{3}{2} + \left(\frac{-3}{2}\right)(x-2)$

$T(x) = 1.5 + (-1.5)(x-2)$

x	1.9	1.99	2	2.01	2.1
$f(x)$	1.6620	1.5151	1.5	1.4851	1.3605
$T(x)$	1.65	1.515	1.5	1.485	1.35

All of the differentiation rules can be written in differential form.

By definition of differentials, we have for functions (of x) u and v:

$du = u'dx$ and $dv = v'dx$

Note that rearranged, these look like $\frac{du}{dx} = u'$ and $\frac{dv}{dx} = v'$.

For example, the Product Rule becomes:

$$d[uv] = [uv]'dx = [uv' + vu']dx = uv'dx + vu'dx = udv + vdu$$

Differential Formulas

Constant multiple:	$d[cu] = cdu$
Sum or difference:	$d[u \pm v] = du \pm dv$
Product:	$d[uv] = udv + vdu$
Quotient:	$d\left[\frac{u}{v}\right] = \frac{vdu - udv}{v^2}$

3.9 #8 $y = 1 - 2x^2 = f(x)$; $c = 0$; $\Delta x = dx = -0.1$

Compare dy and Δy for the given values of x and Δx.

$f(x) = 1 - 2x^2$

$f(0) = 1$

$f'(x) = -4x$

$f'(0) = 0$

$\Delta y = f(c + \Delta x) - f(c)$

$= 1 - 2(0 + (-0.1))^2 - 1$

$= -2(-0.1)^2$

$= -2(0.01)$

$= \boxed{-0.02}$

$dy = f'(c)dx$

$= 0(-0.1)$

$= \boxed{0}$

Find the differential dy.

$dy = f'(x)dx$

12. $y = 3x^{2/3}$

$dy = 2x^{-1/3}dx$

16. $y = \sqrt{x} + \frac{1}{\sqrt{x}} = x^{1/2} + x^{-1/2}$

$dy = \left(\frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2}\right)dx$

$dy = \frac{dx}{2\sqrt{x}} - \frac{dx}{2\sqrt{x^3}}$

14. $y = \sqrt{9 - x^2} = (9-x^2)^{1/2}$

$dy = \frac{1}{2}(9-x^2)^{-1/2}(-2x)dx$

$dy = \frac{-x\,dx}{\sqrt{9-x^2}}$

20. $y = \frac{\sec^2 x}{x^2 + 1} = \frac{(\sec x)^2}{x^2+1}$

$dy = \frac{(x^2+1)(2\sec x \cdot \sec x \tan x) - (\sec^2 x)(2x)}{(x^2+1)^2}dx$

<u>3.9 #46</u>

Use differentials to approximate $\sqrt[3]{26}$ $= 26\text{^}(1/3)$

$\approx 2.96249606...$

$$\left.\begin{aligned} \Delta y &= f(c + \Delta x) - f(c) \\ dy &= f'(x)dx \\ \Delta y &\approx dy \end{aligned}\right\} \rightarrow \boldsymbol{f(c + \Delta x) - f(c) \approx f'(x)dx}$$

$$\boxed{f(c + \Delta x) \approx f(c) + f'(c)dx}$$

$f(x) = \sqrt[3]{x} = x^{1/3}$; $c = 27$; $\Delta x = dx = -1$

$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$

$\sqrt[3]{26} = \sqrt[3]{27 + (-1)} \approx \sqrt[3]{27} + \frac{1}{3\sqrt[3]{27^2}} \cdot (-1)$

$= 3 + \frac{1}{3 \cdot 9}(-1) = 3 - \frac{1}{27} = \frac{80}{27}$

$\approx 2.9\overline{62}$

Recall rules of exponents: $x^{m/n} = (x^m)^{1/n} = \left(x^{1/n}\right)^m$

$= \sqrt[n]{x^m} \quad = \left(\sqrt[n]{x}\right)^m$

3.9 #50

Use differentials to approximate $\tan(0.05)$. ≈ 0.0500417708

calculator yields:

$f(c+\Delta x) \approx f(c) + f'(x)dx$

$f(x) = \tan x$; $c = 0$; $\Delta x = dx = 0.05$

$f'(x) = \sec^2 x$ $f(c) = \tan(0) = 0$

$\tan(0.05) = \tan(0 + 0.05)$

$\approx \tan 0 + [\sec^2(0)](0.05)$

$= 0 + 1(0.05)$

$= \boxed{0.05}$

Homework #9

- **(due Fri, 10/17)**
- 3.5 #15-31odd - limits at infinity
- 3.7 #3,5,17,23,29 - optimization
- 7.7 #11-35odd - l'Hopital's rule
- 7.7 #37-53odd - l'Hopital's rule with logs

Test 4 Thurs. 10/23

Homework #10 (due ~~Fri. 10/24~~) Thurs. 10/23

3.9 #5, 9; 11-19 odd; 45, 49

handout problems due Fri. 10/24

Exam Wed 10/29