1. If $\begin{cases} f(x) = \frac{\sqrt{2x+5}-\sqrt{x+7}}{x-2}, for\ x \neq 2 \\ f(2) = k \end{cases}$ and f is continuous at $x = 2$, then $k =$

(A) 0 (B) $\boxed{\frac{1}{6}}$ (C) $\frac{1}{3}$ (D) 1 (E) $\frac{7}{5}$

Continuity

If $\lim_{x\to c} f(x) = f(c)$, we say that the function f is continuous at c.

Discontinuities occur when either
1. $f(x)$ is undefined,
2. $\lim_{x\to c} f(x)$ does not exist, or
3. $\lim_{x\to c} f(x) \neq f(c)$

If a discontinuity can be removed by inserting a single point, it is called removable. Otherwise, it is nonremovable (e.g. verticals asymptotes and jump discontinuities)

$$\lim_{x\to 2} \frac{\sqrt{2x+5}-\sqrt{x+7}}{x-2} = k$$

$$\lim_{x\to 2} \frac{\sqrt{2x+5}-\sqrt{x+7}}{x-2} \cdot \frac{\sqrt{2x+5}+\sqrt{x+7}}{\sqrt{2x+5}+\sqrt{x+7}}$$

$$= \lim_{x\to 2} \frac{(2x+5)-(x+7)}{(x-2)(\sqrt{2x+5}+\sqrt{x+7})}$$

$$= \lim_{x\to 2} \frac{\cancel{x-2}}{\cancel{(x-2)}(\sqrt{2x+5}+\sqrt{x+7})} = \frac{1}{\sqrt{2(2)+5}+\sqrt{2+7}} = \boxed{\frac{1}{6}}$$

Formal Definition of a Limit

Let f be a function defined on an open interval containing c (except possibly at c) and let L be a real number. The statement $\lim_{x\to c} f(x) = L$ means that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

2. Of the following choices of δ, which is the *largest* that could be used successfully with an arbitrary ε in an epsilon-delta proof of $\lim_{x\to 2}(1 - 3x) = -5$?

(A) $\delta = 3\varepsilon$ (B) $\delta = \varepsilon$ (C) $\delta = \frac{\varepsilon}{2}$ (D) $\delta = \frac{\varepsilon}{4}$ (E) $\delta = \frac{\varepsilon}{5}$

$$|f(x)-L| = |1-3x-(-5)| = |-3x+6|$$

$$= |-3(x-2)| = \frac{3|x-2|}{3} < \frac{\varepsilon}{3}$$

$$|x-2| < \boxed{\varepsilon/3}$$

largest δ that works

Let $\varepsilon > 0$ be given. Take $\delta = \varepsilon/4$.
whenever $|x-2| < \delta$, we have the following:
$|f(x)-L| = |1-3x-(-5)| = 3|x-2| < 3\cdot\delta = 3\cdot\frac{\varepsilon}{4} < \varepsilon$,
i.e. $|x-c| < \delta$ implies that $|f(x)-L| < \varepsilon$, and
hence $\lim_{x\to 2}(1-3x)$ is indeed -5.

3. $\lim_{x \to 5^-} \frac{-2|5-x|}{x-5} =$

(A) 0 (B) 1 (C) -1 $\boxed{\text{(D) } 2}$ (E) -2 (F) *does not exist*

Absolute Value:

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$|5-x| = \begin{cases} 5-x, & 5-x \geq 0 \quad 5 \geq x \longrightarrow x \leq 5 \\ -(5-x) = x-5, & 5-x < 0 \quad 5 < x \longrightarrow x > 5 \end{cases}$$

$$\frac{-2|5-x|}{x-5} = \begin{cases} \frac{-2(5-x)}{x-5}, & x < 5 \\ \frac{-2(-1)(5-x)}{x-5}, & x > 5 \end{cases}$$

$$= \begin{cases} 2, & x < 5 \\ -2, & x > 5 \end{cases}$$

4. $\lim_{x \to 0^-} \frac{1}{x} - \frac{\cos x}{x} =$

$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1} \quad and \quad \boxed{\lim_{x \to 0} \frac{1 - \cos x}{x} = 0}$$

$\boxed{\text{(A) } 0}$ (B) 1 (C) -1 (D) ∞ (E) $-\infty$ (F) *does not exist*

$$= \lim_{x \to 0^-} \frac{1 - \cos x}{x} = 0$$

Continuity

If $\lim_{x \to c} f(x) = f(c)$, we say that the function f is continuous at c.

Discontinuities occur when either
1. $f(x)$ is undefined,
2. $\lim_{x \to c} f(x)$ does not exist, or
3. $\lim_{x \to c} f(x) \neq f(c)$

If a discontinuity can be removed by inserting a single point, it is called removable. Otherwise, it is nonremovable (e.g. verticals asymptotes and jump discontinuities)

5. The function $f(x) = \frac{x^2+2x-24}{x^2+4x-12}$ has a *removable* discontinuity at

(A) 2 (B) -2 (C) 4 (D) -4 (E) 6 $\boxed{\text{(F) } -6}$
(G) none of f's discontinuities are removable

$$f(x) = \frac{\cancel{(x+6)}(x-4)}{\cancel{(x+6)}(x-2)}$$

Continuity

If $\lim_{x \to c} f(x) = f(c)$, we say that the function f is continuous at c.

Discontinuities occur when either
1. $f(x)$ is undefined,
2. $\lim_{x \to c} f(x)$ does not exist, or
3. $\lim_{x \to c} f(x) \neq f(c)$

If a discontinuity can be removed by inserting a single point, it is called removable. Otherwise, it is nonremovable (e.g. verticals asymptotes and jump discontinuities)

6. For the function $f(x) = \begin{cases} -2x^2+3x, & x \leq -3 \\ -5x-6, & -3 < x \leq 2 \\ x^4, & x > 2 \end{cases}$, the discontinuity at $x = -3$ is

(A) removable (hole) ~~(B) non-removable (hole)~~
~~(C) removable (jump)~~ $\boxed{\text{(D) non-removable (jump)}}$
~~(E) removable (vertical asymptote)~~ (F) non-removable (vertical asymptote)

$-2(-3)^2+3(-3)$ $\qquad$ $-5(-3)-6$
$-2(9)-9$ $\qquad$ $15-6$
$=-27$ $\qquad$ $=9$

7. For the function $f(x) = 2x^3 - 3x$, determine $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

(A) 0 (B) -3 (C) $6x^2 - 3$ (D) $2h^2 - 3$ (E) $6x^2 + 6xh + 2h^2 - 3$
(F) *does not exist*

Binomials and the Binomial Theorem

$n! = n(n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot 1$

"n choose k" = $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ = the number of ways to choose k objects out of a set containing n objects

$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$

The $(k+1)^{st}$ *term of* $(a+b)^n$ *is* $\binom{n}{k} a^{n-k} b^k$

Patterns to be found in $(a+b)^n$:

- $(a+b)^n$ has n+1 terms
- powers of a decrease from n to 0
- powers of b increase from 0 to n
- sum of exponents in each term is n
- coefficients come from Pascal's triangle

1	= Degree 0
1 1	= Degree 1
1 2 1	= Degree 2
1 3 3 1	= Degree 3
1 4 6 4 1	= Degree 4
1 5 10 10 5 1	= Degree 5
1 6 15 20 15 6 1	= Degree 6
1 7 21 35 35 21 7 1	= Degree 7
1 8 28 56 70 56 28 8 1	= Degree 8
1 9 36 84 126 126 84 36 9 1	= Degree 9
1 10 45 120 210 252 210 120 45 10 1	= Degree 10

$$(x+h)^3 = \sum_{k=0}^{3} \binom{3}{k} x^{3-k} h^k$$

$$= \binom{3}{0} x^{3-0} h^0 + \binom{3}{1} x^{3-1} h^1 + \binom{3}{2} x^{3-2} h^2 + \binom{3}{3} x^{3-3} h^3$$

$$= \frac{3!}{0!(3-0)!} x^3 h^0 + \frac{3!}{1!(3-1)!} x^2 h^1 + \frac{3!}{2!(3-2)!} x^1 h^2 + \frac{3!}{3!(3-3)!} x^0 h^3$$

$$= 1x^3 + 3x^2h + 3xh^2 + 1h^3$$

$$\lim_{h\to 0} \frac{2(x+h)^3 - 3(x+h) - (2x^3 - 3x)}{h}$$

$$= \lim_{h\to 0} \frac{2(x^3 + 3x^2h + 3xh^2 + h^3) - 3x - 3h - 2x^3 + 3x}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{2x^3} + 6x^2h + 6xh^2 + 2h^3 - \cancel{3x} - 3h - \cancel{2x^3} + \cancel{3x}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(6x^2 + 6xh + 2h^2 - 3)}{\cancel{h}}$$

$$= 6x^2 + 6x(0) + 2(0)^2 - 3 = \boxed{6x^2 - 3}$$

Intermediate Value Theorem

If f is continuous on the closed interval $[a, b]$ and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that $f(c) = k$.

8. In which of the following situations does the Intermediate Value Theorem guarantee a $c \in [a, b]$ such that $f(c) = 6$?

(A) $f(a) = -1$ *and* $f(b) = 1$ (B) $f(a) = -1$ *and* $f(b) = 6$
(C) $f(a) = 6$ *and* $f(b) = 10$ (D) $f(a) = -1$ *and* $f(b) = 10$
(E) *none of these*

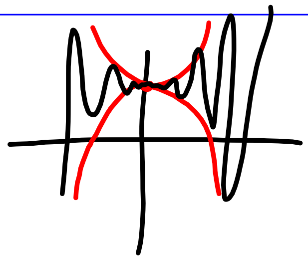
need either $f(a) \le 6$ & $f(b) \ge 6$

or $f(a) \ge 6$ & $f(b) \le 6$

Squeeze Theorem
If $g(x) \le f(x) \le h(x)$, that is, a function is bounded above and below by two other functions, and $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$, that is, the two upper and lower functions have the same limit, then $\lim_{x\to c} f(x) = L$, that is, the limit of the function that is "squeezed" between the two functions with equal limits must have the same limit.

9. $\lim\limits_{x\to 0}\left(-x^2\cos\frac{2}{x}+3\right) =$

(A) 0 (B) -1 (C) 1 $\boxed{\text{(D) } 3}$ (E) -3 (F) *does not exist*



$$-1 \le \cos\frac{2}{x} \le 1$$

$$x^2 \ge -x^2\cos\frac{2}{x} \ge -x^2$$

$$x^2+3 \ge -x^2\cos\frac{2}{x}+3 \ge -x^2+3$$

$$\underbrace{\lim_{x\to 0} x^2+3}_{3} \ge \lim_{x\to 0} -x^2\cos\frac{2}{x}+3 \ge \underbrace{\lim_{x\to 0} -x^2+3}_{3}$$

10. $\lim\limits_{x\to 0}\frac{2\sin 2x}{3x} =$

(A) 0 (B) $\frac{2}{3}$ (C) $\frac{3}{2}$ $\boxed{\text{(D) } \frac{4}{3}}$ (E) $\frac{3}{4}$ (F) *does not exist*

$\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$ *and* $\lim\limits_{x\to 0}\frac{1-\cos x}{x} = 0$

$$\sin 2x \ne 2\sin x$$

$$\sin 2x = 2\sin x\cos x$$

$$\lim_{x\to 0}\frac{2\sin 2x}{3x} = \lim_{x\to 0}\frac{2(2\sin x\cos x)}{3x}$$

$$= \lim_{x\to 0}\frac{4}{3}\cdot\frac{\sin x}{x}\cdot\frac{\cos x}{1}$$

$$\frac{4}{3}\cdot 1 \cdot \underbrace{1}_{\cos(0)} = \frac{4}{3}$$

$$\lim_{x\to 0}\frac{2\sin 2x}{3x} = \lim_{x\to 0}\frac{\sin 2x}{2x}\cdot\frac{2}{3}\cdot\frac{2}{1}$$

$$1 \quad \frac{2}{3} \quad 2 = \boxed{\frac{4}{3}}$$

Continuity

If $\lim_{x \to c} f(x) = f(c)$, we say that the function f is continuous at c.

Discontinuities occur when either
1. $f(x)$ is undefined,
2. $\lim_{x \to c} f(x)$ does not exist, or
3. $\lim_{x \to c} f(x) \neq f(c)$

If a discontinuity can be removed by inserting a single point, it is called removable. Otherwise, it is nonremovable (e.g. verticals asymptotes and jump discontinuities)

Determine the values of b and c such that the function is continuous on the entire real number line.

$$f(x) = \begin{cases} x + 10, & -8 < x < 2 \\ x^2 + bx + c, & |x+3| \geq 5 \end{cases} \longrightarrow$$

$x + 3 \geq 5$
$x \geq 2$

$x + 3 \leq -5$
$x \leq -8$

@ $x = 2$

$2 + 10 = 2^2 + b(2) + c$
$12 = 4 + 2b + c$
$8 = 2b + c$
$c = 8 - 2b$

@ $x = -8$

$-8 + 10 = (-8)^2 + b(-8) + c$
$2 = 64 - 8b + c$
$8b - c = 62$
$c = 8b - 62$

$8 - 2b = 8b - 62$
$-10b = -70$
$\boxed{b = 7}$

$c = 8 - 2(7)$
$= 8 - 14 = \boxed{-6 = c}$