Homework for Test #2 on Derivatives

- 2.1 #1-23 odd Find the derivative by the limit process
- 2.1 #29-32 all find the equation of the tangent line
- 2.1 #61-69 odd Use the alternate form to find the derivative
- 2.1 #71-79 odd Describe x-values where the function is differentiable (given graph)
- 2.2 #3-51 odd Find the derivative using the basic derivative rules
- 2.2 #91-94 all; 101, 102 use the derivative to solve rate of change word problem
- 2.3 #1-53 odd, 63-69 odd, Product and quotient rules
- 2.3 #75-81 all, 83-91 odd, 109-115 all
- 2.4 #7-33 odd, 47-81 odd Chain rule
- 5.1 #45-61, 71 Logarithmic functions
- 5.4 #39-57 Exponential functions
- 5.5 #41-55 Log and exp functions with other bases
- 5.8 #41-59 Inverse trig functions

Fri.

Recommended:
Work through intuitive exercises on Khan Academy:

- Slope of secant lines
- Tangent lope is limiting value of secant slope
- Derivative intuition
- Visualizing derivatives
- Graphs of functions and their derivatives
- The formal and alternate form of the derivative
- Derivatives 1
- Recognizing slopes of curves
- Power rule
- Special derivatives

The Derivative

The slope of the tangent line to the graph of f at the point $(c, f(c))$ is given by:

$$m = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x}$$

The derivative of f at x is given by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Find the equation of the tangent line to

$f(x)=x^3-x$ at the point $(2,6)$.

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f(\overset{2}{c}+h)-\overset{6}{f(c)}}{h}$$

$$m=\lim_{h\to 0}\frac{(2+h)^3-(2+h)-6}{h}$$

$$=\lim_{h\to 0}\frac{2^3+3\cdot 2^2h+3\cdot 2\cdot h^2+h^3-2-h-6}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{8}+12h+6h^2+h^3-h-\cancel{8}}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{h}(12+6h+h^2-1)}{\cancel{h}} = 12-1=\boxed{11}=m$$

point: $(2,6)$ slope: 11

$y-y_1=m(x-x_1)$

$y-6=11(x-2)$

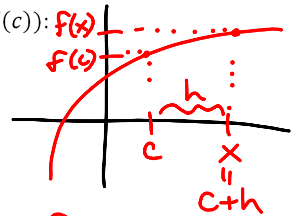
$y=11x-22+6$

$\boxed{y=11x-16}$

2.1 Differentiability & Continuity

Alternative definition of the derivative at the point $(c,f(c))$:

$$f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c} = \lim_{h\to 0}\frac{f(c+h)-f(c)}{\underbrace{c+h-c}_{h}}$$

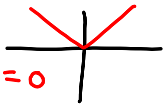


All differentiable functions are continuous, but not all continuous functions are differentiable.

for $f'(c)$ to exist

$$\lim_{x\to c^-}\frac{f(x)-f(c)}{x-c} = \lim_{x\to c^+}\frac{f(x)-f(c)}{x-c}$$

e.g. $f(x)=|x|$



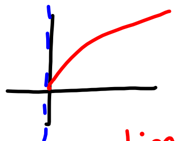
$c=0$

$$\lim_{x\to 0^-}\frac{|x|-|0|}{x-0} = \lim_{x\to 0^-}\frac{|x|}{x} = -1$$

$$\lim_{x\to 0^+}\frac{|x|-|0|}{x-0} = \lim_{x\to 0^+}\frac{|x|}{x} = 1$$

left- & right-hand limits are different, so the limit in general is undefined & hence the derivative of $|x|$ at $x=0$ does not exist.

$f(x) = \sqrt{x}$ continuous on $[0, \infty)$



$\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$ @ $c = 0$

$f'(0) = \lim_{x \to 0^+} \frac{\sqrt{x} - \sqrt{0}}{x - 0} = \lim_{x \to 0^+} \frac{\sqrt{x}}{x} = \lim_{x \to 0^+} \frac{x^{1/2}}{x^1}$

$= \lim_{x \to 0^+} \frac{1}{x^{1-1/2}} = \lim_{x \to 0^+} \frac{1}{x^{1/2}}$

$= \lim_{x \to 0^+} \frac{1}{\sqrt{x}} \to \underline{+\infty}$

$\frac{x^m}{x^n} = x^{m-n} = \frac{1}{x^{n-m}}$

$x^m x^n = x^{m+n}$

$(x^m)^n = x^{m+n}$

$\Rightarrow$ vertical tangent line (undefined slope)

derivative does not exist

2.2 Basic Differentiation Rules

1. The derivative of a constant function is zero, i.e.,

for $c \in \mathbb{R}$, $\frac{d}{dx}[c] = 0$

Proof: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$[c]' = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} \frac{0}{h} = \lim_{h \to 0} 0 = \boxed{0}$

2. Power Rule for $n \in \mathbb{Q}$, $\frac{d}{dx}[x^n] = nx^{n-1}$ Special case: $\frac{d}{dx}[x] = 1$

Proof:

Recall the binomial expansion:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \cdots + \frac{n!}{k!\,(n-k)!}a^{n-k}b^k + \cdots + b^n$$

$$[x^n]' = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^n} + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + h^n - \cancel{x^n}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}\left(nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}\cdot h + \cdots + h^{n-1}\right)}{\cancel{h}}$$

$$= \boxed{nx^{n-1}}$$

Examples:

$\frac{d}{dx}[x^7] = 7x^6$

$\frac{d}{dx}[\pi^3] = 0$

$\frac{d}{dx}[2e] = 0$

$\frac{d}{dx}[\sqrt{x}] = \frac{d}{dx}[x^{1/2}] = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-1/2} = \boxed{\frac{1}{2\sqrt{x}}}$

$\frac{d}{dx}\left[\frac{1}{x^3}\right] = \frac{d}{dx}[x^{-3}] = -3x^{-3-1} = -3x^{-4} = \boxed{\frac{-3}{x^4}}$

2.2 Basic Differentiation Rules

1. The derivative of a constant function is zero, i.e., for $c \in \mathbb{R}$, $\frac{d}{dx}[c] = 0$

special case: $\frac{d}{dx}[x] = 1$

2. Power Rule for $n \epsilon \mathbb{Q}$, $\frac{d}{dx}[x^n] = nx^{n-1}$

$\frac{d}{dx}[x^1] = 1 \cdot x^0 = 1 \cdot 1 = 1$

3. Constant Multiple Rule $\in \mathbb{R}$, $\frac{d}{dx}[cf(x)] = cf'(x)$

$\frac{d}{dx}[cx] = c$

4. Sum & Difference Rules $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$

Examples:

$f(x) = 3x^2$

$f'(x) = (3x^2)' = 3(x^2)' = 3(2x) = \boxed{6x}$

$f(x) = \frac{3}{x} = 3x^{-1}$

$f'(x) = 3(-x^{-2}) = \boxed{\frac{-3}{x^2}}$

$g(x) = 2x^3 - x^2 + 3x$

$g'(x) = \boxed{6x^2 - 2x + 3}$

$y = 4x^{3/2} - 5x^4 + 2x^{\frac{1}{3}} - 7$

$y' = \boxed{6x^{1/2} - 20x^3 + \frac{2}{3}x^{-2/3}}$

Derivatives of Trig Functions

1. $\frac{d}{dx}[\sin x] = \cos x$
2. $\frac{d}{dx}[\cos x] = -\sin x$
3. $\frac{d}{dx}[\tan x] = \sec^2 x$
4. $\frac{d}{dx}[\cot x] = -\csc^2 x$
5. $\frac{d}{dx}[\sec x] = \sec x \tan x$
6. $\frac{d}{dx}[\csc x] = -\csc x \cot x$

Proof that (sin x)'=cos x

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$(\sin x)' = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\cos x \sin h \quad - \sin x (1 - \cos h)}{h}$$

$$= \lim_{h\to 0} \frac{\cos x}{1} \cdot \frac{\sin h}{h} - \lim_{h\to 0} \frac{\sin x}{1} \cdot \frac{1 - \cos h}{h}$$

$$\cos x \cdot 1 - \sin x \cdot 0$$

$$= \boxed{\cos x}$$

The Derivative

The slope of the tangent line to the graph of f at the point $(c, f(c))$ is given by:

$$m = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x}$$

The derivative of f at x is given by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

2.2 Basic Differentiation Rules

1. The derivative of a constant function is zero, i.e., for $c \in \mathbb{R}$, $\frac{d}{dx}[c] = 0$

2. Power Rule for $n \epsilon \mathbb{Q}$, $\frac{d}{dx}[x^n] = nx^{n-1}$

3. Constant Multiple Rule $\in \mathbb{R}$, $\frac{d}{dx}[cf(x)] = cf'(x)$

4. Sum & Difference Rules $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$

Derivatives of Trig Functions

1. $\frac{d}{dx}[\sin x] = \cos x$
2. $\frac{d}{dx}[\cos x] = -\sin x$
3. $\frac{d}{dx}[\tan x] = \sec^2 x$
4. $\frac{d}{dx}[\cot x] = -\csc^2 x$
5. $\frac{d}{dx}[\sec x] = \sec x \tan x$
6. $\frac{d}{dx}[\csc x] = -\csc x \cot x$

2.2

22. $y = 5 + \sin x$

$y' = (5)' + (\sin x)' = 0 + \cos x = \boxed{\cos x}$

24. $y = \frac{5}{(2x)^3} + 2\cos x = \frac{5}{8}x^{-3} + 2\cos x$

$y' = \boxed{\frac{-15}{8}x^{-4} - 2\sin x}$

44. $h(x) = \dfrac{2x^3 - 3x + 1}{x} = \dfrac{2x^3}{x} - \dfrac{3x}{x} + \dfrac{1}{x} = 2x^2 - 3 + x^{-1}$

$h'(x) = 4x - 0 - x^{-2} = \boxed{4x - x^{-2}}$

$4x - \dfrac{1}{x^2} = \dfrac{4x^3}{x^2} - \dfrac{1}{x^2}$

$= \dfrac{4x^3 - 1}{x^2}$

46. $y = 3x(6x - 5x^2)$

$= 18x^2 - 15x^3$

$y' = \boxed{36x - 45x^2}$