Homework #15 - due Fri, 10/16

- 3.5 #15-31odd - limits at infinity
- 3.7 #3,5,17,23,29 - optimization

Homework #16 - due Wed, 10/21

- 7.7 #11-35odd - l'Hopital's rule
- 7.7 #37-53odd - l'Hopital's rule with logs

Test #4 - Fri 23

1. Find y' in terms of x and y.

$x^3y + y^3 - 3xy = 4$

$3x^2y + \underline{x^3y'} + \underline{3y^2y'} - 3y - \underline{3xy'} = 0$

$y'(x^3 + 3y^2 - 3x) = 3y - 3x^2y$

$$\boxed{y' = \frac{3y - 3x^2y}{x^3 + 3y^2 - 3x}}$$

2. Find y' in terms of x and y.

$y = \cos(xy) - x$

$y' = [-\sin(xy)](1 \cdot y + xy') - 1$

$y' = -y\sin(xy) - xy'\sin(xy) - 1$

$y' + xy'\sin(xy) = -y\sin(xy) - 1$

$y'(1 + x\sin(xy)) = -y\sin(xy) - 1$

$$\boxed{y' = \frac{-y\sin(xy) - 1}{1 + x\sin(xy)}}$$

3. Find y'' in terms of x and y.

$3 - 5y^2 = 4x - 2y$

$-10y \cdot y' = 4 - 2y'$

$2y' - 10yy' = 4$

$y'(2 - 10y) = 4$

$y = \frac{4}{2 - 10y} = \frac{\cancel{2} \cdot 2}{\cancel{2}(1 - 5y)}$

$y' = \frac{2}{1 - 5y} =$

$= 2(1 - 5y)^{-1}$

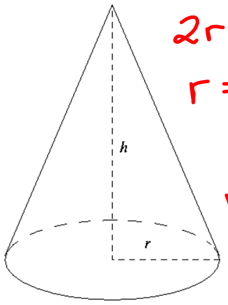
$y'' = -2(1 - 5y)^{-2}(-5y')$

$= \frac{10}{(1 - 5y)^2} \cdot \frac{2}{1 - 5y}$

$$\boxed{y'' = \frac{20}{(1 - 5y^3)}}$$

$y'' = \frac{160}{(2 - 10y)^3}$

4. Soft-serve ice cream extrudes into a conical pile at a rate of 4 cubic inches per second. The diameter of the base of the cone is approximately five times the height. Given that the volume of a cone is $V = \frac{1}{3}\pi r^2 h$, how fast is the height of the ice cream changing when the pile is 6 inches tall?



$2r = 5h$

$r = \frac{5h}{2}$

$\frac{dV}{dt} = 4 \text{ in}^3/\text{s}$

$\frac{dh}{dt} = ?$ when $h = 6$ in

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi \left(\frac{5h}{2}\right)^2 \cdot h$

$V = \frac{1}{3}\pi \cdot \frac{25h^2}{4} \cdot h$

$\star$ $V = \frac{25\pi}{12} h^3$

$\frac{dV}{dt} = \frac{25\pi}{\cancel{12}\,4} \cdot \cancel{3}h^2 \cdot \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{\frac{dV}{dt}}{\frac{25\pi}{4}h^2} = \frac{4}{\frac{25\pi}{4}(6)^2} = \boxed{\frac{4}{225\pi} \text{ in/s}}$

≈ 0.005659

5. When a guitar string is plucked, it vibrates with a frequency of $F = 200\sqrt{T}$, where F is measured in vibrations per second and the tension T is measured in pounds.

a. Find the *instantaneous* rate of change of F when $T = 4$. Give an exact answer. Units should be vibrations per second per pound.

$F = 200T^{1/2}$

$F' = 100T^{-1/2} = \frac{100}{\sqrt{T}}\Big|_{T=4} = \frac{100}{\sqrt{4}} = \frac{100}{2} = \boxed{50}$

b. Find the *average* rate of change of F as the tension changes from $T = 1$ to $T = 9$. Give an exact answer. Units should be vibrations per second per pound.

$\frac{200\sqrt{9} - 200\sqrt{1}}{9-1} = \frac{600-200}{8} = \boxed{50}$

Bonus A : Find the equation of tangent line to the graph of the equation at the indicated point.

$$x^2 - y^3 = 6\ , \qquad (5, 2)$$

$2x - 3y^2 y' = 0$

$-3y^2 y' = -2x$

$y' = \frac{-2x}{-3y^2} = \frac{2x}{3y^2}$

$m = \frac{2(5)}{3(2)^2} = \frac{5}{6}$

$y - y_1 = m(x - x_1)$

$y - 2 = \frac{5}{6}(x - 5)$

$y - \frac{12}{6} = \frac{5}{6}x - \frac{25}{6}$

$\boxed{y = \frac{5}{6}x - \frac{13}{6}}$

Bonus B: Find $f'(x)$ for $f(x) = \log_4\left[\tan\left(2^{(7x^2-5x)}\right)\right]$

$f'(x) = \frac{1}{\ln 4 \cdot \tan\left(2^{7x^2-5x}\right)} \cdot \sec^2\left(2^{7x^2-5x}\right) \cdot 2^{7x^2-5x} \cdot \ln 2 \cdot (14x - 5)$

Determine if the Mean Value Theorem can be applied to f on the closed interval $[a, b]$, and if so, find all values of c in the open interval (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

$$f(x) = \frac{x+1}{x} \quad ; \quad \left[\frac{1}{2}, 2\right]$$

f cts on $\left[\frac{1}{2}, 2\right]$? yes
f diff on $\left(\frac{1}{2}, 2\right)$? yes } MVT applies

$$\frac{f(2)-f(\frac{1}{2})}{2-\frac{1}{2}} = \frac{\frac{2+1}{2} - \frac{\frac{1}{2}+1}{\frac{1}{2}}}{2-\frac{1}{2}} = -1$$

$$f'(x) = \frac{x(1)-(x+1)(1)}{x^2} = \frac{-1}{x^2}$$

$$-\frac{1}{x^2} = -1$$

$$x^2 = 1$$

$$x = \pm 1$$

$$\boxed{x = 1}$$

Determine if Rolle's Theorem can be applied to f on the closed interval $[a, b]$. If Rolle's Theorem can be applied, find all values of c in the open interval (a, b) such that $f'(c) = 0$.

$$f(x) = x^3 - 5x^2 - 8x + 3 \quad ; \quad [-2,4]$$

f cts on $[-2,4]$? yes
f diff on $(-2,4)$? yes
$f(a) = f(b)$? NO
$f(-2) = -9$
$f(4) = -45$

Rolle's Thm does not apply!

Locate the absolute extrema of the function on the closed interval.

$$f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x + 4\,;\ [-1,2]$$

$f'(x) = x^2 + x - 2$

$(x+2)(x-1) = 0$

$\cancel{x=-2}$, $x = 1$

$f(-1) = 37/6 \leftarrow$ abs max

$f(1) = 17/6 \leftarrow$ abs min

$f(2) = 28/6$

Find the open intervals on which the function is increasing or decreasing and locate all relative extrema.

$$f(x) = \frac{x^2+3}{x+1}$$

$$f'(x) = \frac{(x+1)(2x) - (x^2+3)(1)}{(x+1)^2} = \frac{2x^2+2x-x^2-3}{(x+1)^2}$$

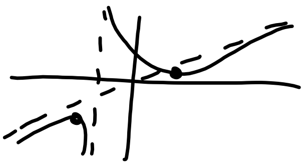
$$= \frac{x^2+2x-3}{(x+1)^2} = \frac{(x+3)(x-1)}{(x+1)^2}$$

critical #'s: $-3, -1, 1$

$f'(-4)$ $f'(-2)$ $f'(0)$ $f'(2)$

$+$ -3 $-$ -1 $-$ 1 $+$

$\nearrow$ $\searrow$ $\searrow$ $\nearrow$



f is increasing on $(-\infty,-3) \cup (1,\infty)$

f is decreasing on $(-3,-1) \cup (-1,1)$

relative max @ $(-3,-6)$

relative min @ $(1,2)$

Find the points of inflection and discuss concavity of the graph of the function.

$$f(x) = x^4 - 18x^2 + 4x - 12$$

$f'(x) = 4x^3 - 36x + 4$

$f''(x) = 12x^2 - 36$

$12(x^2 - 3) = 0$

$x^2 = 3$

$x = \pm\sqrt{3}$

$f''(-4)$, $f''(0)$, $f''(4)$

$+$ $-\sqrt{3}$ $-$ $\sqrt{3}$ $+$

$\cup$ $\cap$ $\cup$

f is concave up on $(-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$

concave down on $(-\sqrt{3}, \sqrt{3})$

inflection points:

$(-\sqrt{3}, \underbrace{-57 - 4\sqrt{3}}_{\approx -64})$, $(\sqrt{3}, \underbrace{-57 + 4\sqrt{3}}_{\approx -50})$

Bonus: Find the polynomial $P_2(x) = a_0 + a_1x + a_2x^2$ whose value and first two derivatives agree with the value and first two derivatives of $f(x) = \cos x$ at the point $(0, a_0)$. This polynomial is called the second-degree Taylor polynomial of $f(x) = \cos x$ at $x = 0$. *[Hint: to find the polynomial means to find the coefficients $a_0, a_1, \& a_2$.]*

$P_2(0) = a_0 + a_1(0) + a_2(0)^2 = f(0) = \cos 0$ $a_0 = 1$

$P_2'(0) = a_1 + 2a_2(0) = f'(0) = -\sin(0)$ $a_1 = 0$

$P_2''(0) = 2a_2 = f''(0) = -\cos(0)$ $a_2 = -\frac{1}{2}$

$$\boxed{P_2(x) = 1 - \frac{1}{2}x^2}$$

7.7 Indeterminate Forms & L'Hôpital's Rule

$\boxed{\frac{0}{0}, \frac{\infty}{\infty},}$ $0 \cdot \infty$, 1^{∞}, 0^0, *and* $\infty - \infty$ are called indeterminate forms.

L'Hôpital's Rule:

Let f and g be functions that are differentiable on an open interval (a, b) containing c, except possibly at c itself. Assume that $g'(x) \neq 0$ for all x in (a, b), except possibly at c itself. If the limit of f(x)/g(x) as x approaches c produces an indeterminate form $0/0$, ∞/∞, $(-\infty)/\infty$, or $\infty/(-\infty)$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

42. $\lim_{x \to 0^+} (e^x + x)^{2/x} = 1^{\infty}$

$\ln(a^b) = b \ln a$

$y = \lim_{x \to 0^+} (e^x + x)^{2/x}$

$\ln y = \ln\left[\lim_{x \to 0^+} (e^x + x)^{2/x}\right]$

$\ln y = \lim_{x \to 0^+} \left[\ln(e^x + x)^{2/x}\right]$

$\ln y = \lim_{x \to 0^+} \frac{2}{x} \ln(e^x + x)$

$\ln y = \lim_{x \to 0^+} \frac{2\ln(e^x + x)}{x} = \frac{0}{0}$ L'H applies

$\ln y = \lim_{x \to 0^+} \frac{2 \cdot \frac{1}{e^x + x} \cdot (e^x + 1)}{1}$

$\ln y = \lim_{x \to 0^+} \frac{2(e^x + 1)}{e^x + x} = \frac{2(e^0 + 1)}{e^0 + 0} = \frac{2(1+1)}{1+0} = 4$

$\ln y = 4$

$e^{\ln y} = e^4$

$\boxed{y = e^4}$

$\log_a(a^x) = x$

$a^{\log_a x} = x$

$$\lim_{x \to \infty} \frac{x}{\sqrt{x^2}} = \lim_{x \to \infty} \frac{x}{|x|}$$

$$= \lim_{x \to \infty} \frac{x}{x} = \lim_{x \to \infty} 1 = 1$$