The Squeeze Theorem:

If $f(x) \le g(x) \le h(x)$ *and* $\lim_{x\to c} f(x) = L = \lim_{x\to c} h(x)$,

If $f(x) \le g(x)$ AND $g(x) \le h(x)$

Then $\lim_{x\to c} g(x) = L$.

Special Limits Derived by Squeeze Theorem:

$$\boxed{\lim_{x\to 0}\frac{\sin x}{x} = 1} \quad ; \quad \boxed{\lim_{x\to 0}\frac{1-\cos x}{x} = 0}$$

Memorize!!

Use the squeeze theorem to find

$$\lim_{x\to 0}\left(x^2\cos\frac{5}{x} - 3\right)$$

$$-1 \le \cos\theta \le 1$$

$$-1 \le \cos\frac{5}{x} \le 1$$

$$-x^2 \le x^2\cos\frac{5}{x} \le x^2$$

$$-x^2-3 \le x^2\cos\frac{5}{x}-3 \le x^2-3$$

$$\lim_{x\to 0}(-x^2-3) \le \lim_{x\to 0}\left(x^2\cos\frac{5}{x}-3\right) \le \lim_{x\to 0}(x^2-3)$$

$$-3 \le \lim_{x\to 0}\left(x^2\cos\frac{5}{x}-3\right) \le -3$$

$\Rightarrow$ By the Squeeze Theorem $\lim_{x\to 0}\left(x^2\cos\frac{5}{x}-3\right) = \boxed{-3}$

68. $\lim_{x\to 0} \frac{3(1-\cos x)}{x}$

$= 3 \cdot \lim_{x\to 0} \frac{1-\cos x}{x}$

$= 3 \cdot 0 = \boxed{0}$

$\lim_{x\to 0} \frac{\sin x}{x} = 1$

$\lim_{x\to 0} \frac{1-\cos x}{x} = 0$

72. $\lim_{x\to 0} \frac{\tan^2 x}{x} = \lim_{x\to 0} \frac{\sin^2 x}{x \cos^2 x}$

$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

$= \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{\cos^2 x}$

$= \left(\lim_{x\to 0} \frac{\sin x}{x}\right)\left(\lim_{x\to 0} \frac{\sin x}{\cos^2 x}\right)$

$= 1 \cdot \frac{\sin 0}{\cos^2 0} = 1 \cdot 0 = \boxed{0}$

78. $\lim_{x\to 0} \dfrac{\sin 2x}{\sin 3x}$

$$= \lim_{x\to 0} \frac{\frac{\sin 2x}{1}\cdot\frac{2x}{2x}}{\frac{\sin 3x}{1}\cdot\frac{3x}{3x}}$$

$$= \lim_{x\to 0} \frac{\frac{\sin 2x}{2x}\cdot\frac{2\cancel{x}}{1}}{\frac{\sin 3x}{3x}\cdot\frac{3\cancel{x}}{1}}$$

$$= \frac{\left(\lim_{x\to 0}\frac{\sin 2x}{2x}\right)\cdot 2}{\left(\lim_{x\to 0}\frac{\sin 3x}{3x}\right)\cdot 3} = \frac{1\cdot 2}{1\cdot 3} = \boxed{\frac{2}{3}}$$

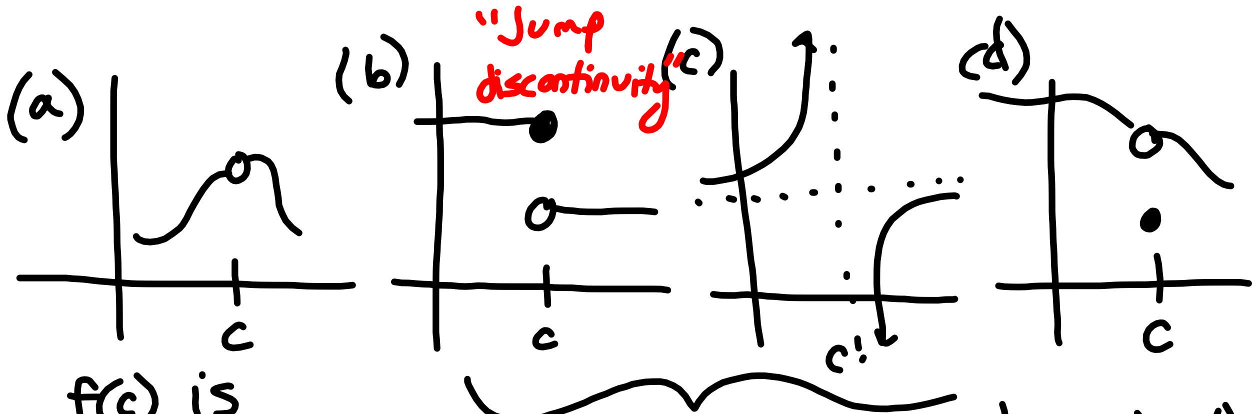
$\lim_{x\to 0}\frac{\sin x}{x} = 1$

$\lim_{x\to 0}\frac{\sin 2x}{2x} = 1$

$\lim_{x\to 0}\frac{\sin 3x}{3x} = 1$

$\frac{a}{1}\cdot\frac{b}{c} = \frac{ab}{c} = \frac{a}{c}$!

1.4 Continuity and One-Sided Limits

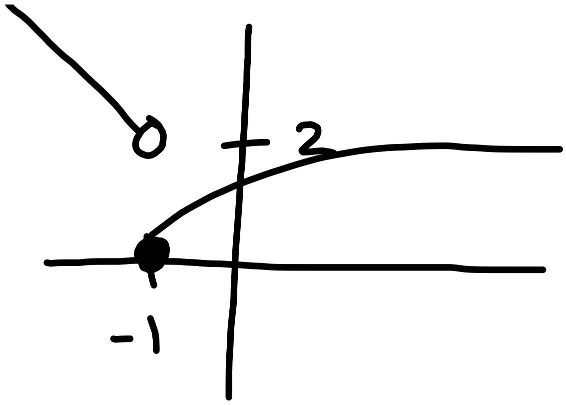


f(c) is undefined

$\lim_{x\to c} f(x)$ does not exist

$\lim_{x\to c} f(x) \neq f(c)$

These are all discontinuities
(a) and (d) are removable
(b) and (c) are nonremovable



$$\lim_{x\to -1^-} f(x) = 2$$

$$\lim_{x\to -1^+} f(x) = 0$$

$$\lim_{x\to -1} f(x) = \text{DNE}$$

One-Sided Limits

$\lim_{x\to c^+} f(x) = L$ *limit from the right*

$\lim_{x\to c^-} f(x) = L$ *limit from the left*

$$\boxed{\lim_{x\to c} f(x) = L \text{ if and only if } \lim_{x\to c^-} f(x) = L = \lim_{x\to c^+} f(x)}$$

Continuity at a point

A function f is <u>continuous at c</u> if the following 3 conditions are met:

1. $f(c)$ is defined
2. Limit of $f(x)$ exists when x approaches c
3. Limit of $f(x)$ when x approaches c is equal to $f(c)$

f(x) is continuous at c if

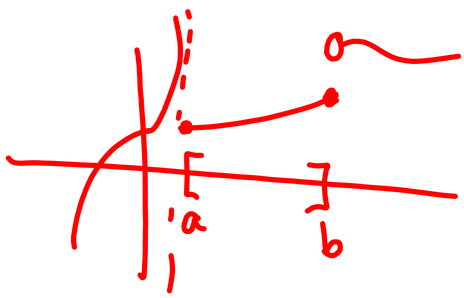
$$\lim_{x\to c} f(x) = f(c)$$

Continuity on an open interval

A function is <u>continuous on an open interval</u> if it is continuous at each point in the interval. A function that is continuous on the entire real line $(-\infty, \infty)$is <u>everywhere continuous</u>.

Continuity on a closed interval

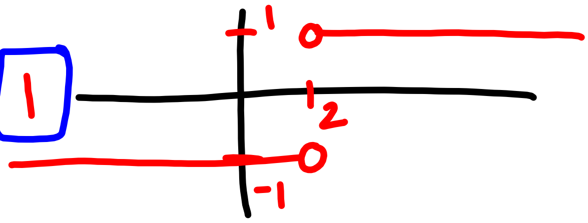
A function f is <u>continuous on the closed interval</u> $[a, b]$ if it is continuous on the open interval I (a, b) and $\lim_{x\to a^+} f(x) = f(a)$ and $\lim_{x\to b^-} f(x) = f(b)$.



10. $\displaystyle \lim_{x\to 4^-}\frac{\sqrt{x}-2}{x-4} = \lim_{x\to 4^-}\frac{\cancel{\sqrt{x}-2}^{\,1}}{(\cancel{\sqrt{x}-2})(\sqrt{x}+2)}$

$\displaystyle = \frac{1}{\sqrt{4}+2} = \boxed{\frac{1}{4}}$

12. $\displaystyle \lim_{x\to 2^+}\frac{|x-2|}{x-2} = \boxed{1}$



$$\frac{|x-2|}{x-2} = \begin{cases} \dfrac{x-2}{x-2} = 1, & x-2>0 \\ & x>2 \\ \dfrac{-(x-2)}{x-2} = -1, & x-2<0 \\ & x>2 \end{cases} = \begin{cases} 1, & x>2 \\ -1, & x<2 \end{cases}$$

$$f(x) = \begin{cases} x^2-2, & x \geq 1 \\ 5, & x < 1 \end{cases}$$

$$\lim_{x \to 1^-} f(x) = 5$$

$$\lim_{x \to 1^+} f(x) = 1^2 - 2 = -1$$

$$\lim_{x \to 1} f(x) = DNE$$