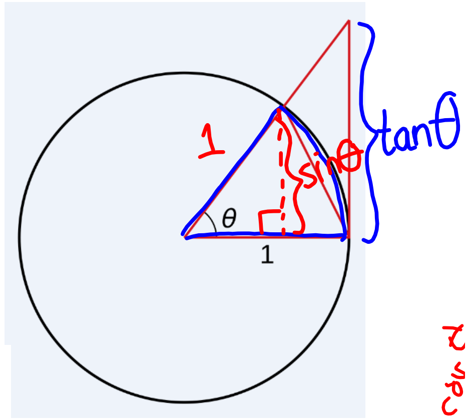
1.3 The Squeeze Theorem



$$\lim_{x\to 0}\frac{\sin x}{x} = ?$$

Area of whole circle $= \pi r^2|_{r=1} = \pi$

$$\frac{\text{Area of whole circle}}{\text{Total angle of circle}} = \frac{\text{Area of sector}}{\theta}$$

$$\frac{\pi}{2\pi} = \frac{\text{Area of sector}}{\theta} \;\rightarrow\; \text{Area of sector} = \frac{\theta}{2}$$

Area of outer triangle $\geq$ Area of sector $\geq$ Area of inner triangle

$$\frac{\tan\theta}{2} \quad\geq\quad \frac{\theta}{2} \quad\geq\quad \frac{\sin\theta}{2}$$

Multiply through by $\frac{2}{\sin\theta}$

$$\frac{\sin\theta}{2\cos\theta}\cdot\frac{2}{\sin\theta} \quad\geq\quad \frac{\theta}{2}\cdot\frac{2}{\sin\theta} \quad\geq\quad \frac{\sin\theta}{2}\cdot\frac{2}{\sin\theta}$$

$$\frac{1}{\cos\theta} \quad\geq\quad \frac{\theta}{\sin\theta} \quad\geq\quad 1$$

$$\sin\theta\cos\theta \cdot \frac{1}{\cos\theta} \geq \frac{\theta}{\sin\theta}\cdot \sin\theta\cos\theta$$

$$\frac{\sin\theta}{\theta} \geq \frac{\theta\cos\theta}{\theta}$$

$$\frac{\sin\theta}{\theta} \geq \cos\theta$$

Take reciprocals and reverse inequalities

$$\cos\theta \quad\leq\quad \frac{\sin\theta}{\theta} \quad\leq\quad 1$$

Take limits

$$\lim_{\theta\to 0}\cos\theta \quad\leq\quad \lim_{\theta\to 0}\frac{\sin\theta}{\theta} \quad\leq\quad \lim_{\theta\to 0} 1$$

$$\cos 0 \leq \lim_{\theta\to 0}\frac{\sin\theta}{\theta} \leq \lim_{\theta\to 0} 1$$

$$1 \leq \lim_{\theta\to 0}\frac{\sin\theta}{\theta} \leq 1$$

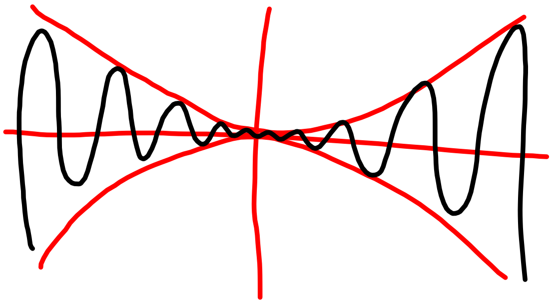
$$\Rightarrow \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

The Squeeze Theorem:

If $f(x) \le g(x) \le h(x)$ *and* $\lim_{x\to c} f(x) = L = \lim_{x\to c} h(x)$,

Then $\lim_{x\to c} g(x) = L$.

$-x^2 \le x^2 \sin x \le x^2$



Special Limits Derived by Squeeze Theorem:

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1} \;;\; \boxed{\lim_{x\to 0} \frac{1-\cos x}{x} = 0}$$

Memorize!!

Use the squeeze theorem to find

$$\lim_{x\to 0}\left(x^2 \cos\frac{5}{x} - 3\right)$$

$-1 \le \cos\theta \le 1$

$-1 \le \cos\frac{5}{x} \le 1$

$a \le b$

$a-3 \le b-3$

$-x^2 \le x^2 \cos\frac{5}{x} \le x^2$

$-x^2-3 \le x^2\cos\frac{5}{x} - 3 \le x^2-3$

$\lim_{x\to 0}(-x^2-3) \le \lim_{x\to 0} x^2\cos\frac{5}{x} - 3 \le \lim_{x\to 0}(x^2-3)$

$-3 \le \lim_{x\to 0} x^2\cos\frac{5}{x} - 3 \le -3$

$\Rightarrow \lim_{x\to 0} x^2\cos\frac{5}{x} - 3 = \boxed{-3}$

68. $\lim\limits_{x\to 0} \dfrac{3(1-\cos x)}{x}$

$$\lim_{x\to a} cf(x) = c \cdot \lim_{x\to a} f(x)$$

$$= 3 \cdot \lim_{x\to 0} \frac{1-\cos x}{x}$$

$$= 3 \cdot 0$$

$$= \boxed{0}$$

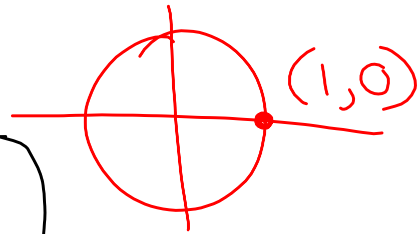
72. $\lim\limits_{x\to 0} \dfrac{\tan^2 x}{x}$

$$= \lim_{x\to 0} \frac{\left(\frac{\sin^2 x}{\cos^2 x}\right)}{\left(\frac{x}{1}\right)} = \lim_{x\to 0} \frac{\sin^2 x}{\cos^2 x} \cdot \frac{1}{x}$$

$$= \lim_{x\to 0} \frac{\sin^2 x}{x\cos^2 x} = \lim_{x\to 0} \left(\frac{\sin x}{x}\right)\left(\frac{\sin x}{\cos^2 x}\right)$$

$$= \lim_{x\to 0} \left(\frac{\sin x}{x}\right) \cdot \lim_{x\to 0} \frac{\sin x}{\cos^2 x}$$

$$= 1 \cdot \frac{\sin 0}{\cos^2 0}$$



$$= 1 \cdot \frac{0}{1} = \boxed{0}$$

78. $\lim_{x \to 0} \dfrac{\sin 2x \cdot \frac{2x}{2x}}{\sin 3x \cdot \frac{3x}{3x}}$

$\sin 2x \neq 2\sin x$ ☹

If $x \to 0$, $2x \to 0$ & $3x \to 0$

$\frac{\sin x}{x} \to 1$ if $x \to 0$

$\frac{\sin(2x)}{2x} \to 1$ if $x \to 0$

$$= \lim_{x \to 0} \frac{\frac{\sin 2x}{2x} \cdot \frac{2\cancel{x}}{1}}{\frac{\sin 3x}{3x} \cdot \frac{3\cancel{x}}{1}}$$

$$= \frac{\lim_{x \to 0}\left(\frac{\sin 2x}{2x}\right)}{\lim_{x \to 0}\left(\frac{\sin 3x}{3x}\right)} \cdot \lim_{x \to 0}\left(\frac{2}{3}\right)$$

$$= \frac{1}{1} \cdot \frac{2}{3} = \boxed{\frac{2}{3}}$$