## 4.4

### Mean Value Theorem for Integrals

If $f$ is cts. $[a,b]$, $\exists c \in [a,b]$ s.t. $\int_a^b f(x)dx = f(c)(b-a)$

46. $f(x) = \frac{9}{x^3}$, $[1,3]$

f cts on $[1,3]$? yes

$$\int_1^3 \frac{9}{x^3}dx = \frac{9}{c^3}(3-1)$$

$$\int_1^3 9x^{-3}dx = \frac{9}{c^3}(2)$$

$$\left.-\frac{9}{2}x^{-2}\right|_1^3 = \frac{18}{c^3}$$

$$\frac{-9}{2(3^2)} - \left(\frac{-9}{2(1)^2}\right) = \frac{18}{c^3}$$

$$-\frac{1}{2} + \frac{9}{2} = \frac{18}{c^3}$$

$$4 = 18/c^3$$

$$c^3 = 18/4$$

$$c = \boxed{\sqrt[3]{18/4} \approx 1.65}$$

### Average Value of a Function on an Interval

If $f$ is integrable on $[a,b]$, then the average value of $f$ on $[a,b]$ is

$$\frac{1}{b-a}\int_a^b f(x)dx$$

50. $f(x) = \frac{4(x^2+1)}{x^2}$, $[1,3]$

avg. value: $\frac{1}{3-1}\int_1^3 (4+4x^{-2})dx$

$$= \frac{1}{2}\left[4x - 4x^{-1}\right]_1^3 = \frac{1}{2}\left[\left(4(3) - \frac{4}{3}\right) - \left(4(1) - \frac{4}{1}\right)\right]$$

$$= \frac{1}{2}\left(12 - \frac{4}{3}\right) = 6 - \frac{2}{3} = \boxed{\frac{16}{3} \approx 5.3}$$

$$\frac{4x^2+4}{x^2} = \frac{16}{3}$$

$$12x^2 + 12 = 16x^2$$

$$0 = 4x^2 - 12$$

$$0 = 4(x^2 - 3)$$

$x = \pm\sqrt{3}$

$-\sqrt{3}$ is not in $[1,3]$

$$\boxed{x = \sqrt{3}}$$

## The Second Fundamental Theorem of Calculus

If $f$ is continuous on an open interval $I$ containing $a$, then for every $x$ in $I$,

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

"Fix x"

76. $F(x) = \int_0^x t(t^2+1)\,dt = \int_0^x (t^3+t)\,dt$

rewrite $F(x)$ as a function of $x$.

$$\left[\frac{t^4}{4}+\frac{t^2}{2}\right]_0^x = \frac{x^4}{4}+\frac{x^2}{2} - \left(\frac{0^4}{4}+\frac{0^2}{2}\right)$$

verify using 2nd FTC.

$$\int_0^x t(t^2+1)\,dt = \frac{x^4}{4}+\frac{x^2}{2}$$

$$\frac{d}{dx}\int_0^x t(t^2+1)\,dt = \frac{d}{dx}\left(\frac{x^4}{4}+\frac{x^2}{2}\right)$$

$$x(x^2+1) = x^3+x \ \checkmark$$

80. $\int_{\pi/3}^x \sec t \tan t\,dt = \sec t \Big|_{\pi/3}^x$

$= \sec x - \sec\frac{\pi}{3}$

$= \sec x - 2$

$$\frac{d}{dx}\int_{\pi/3}^x \sec t\tan t\,dt = \frac{d}{dx}(\sec x - 2)$$

$$\sec x\tan x = \sec x \tan x \ \checkmark$$

86. $F(x) = \int_0^x \sec^3 t \, dt$

$F'(x) = \boxed{\sec^3 x}$

What about $F'(x)$ when $F(x) = \int_a^{g(x)} f(t)\,dt$ ?

Let $g(x) = u$

$$F'(x) = \frac{dF}{dx} = \frac{dF}{du} \cdot \frac{du}{dx} = \frac{d}{du}\left[F\right] \cdot \frac{du}{dx}$$

$$= \frac{d}{du}\left[\int_a^u f(t)\,dt\right] \cdot \frac{du}{dx} = f(u) \cdot \frac{du}{dx}$$

(i.e. we have chain rule)

90. $F(x) = \int_2^{x^2} \frac{1}{t^3}\,dt = \int_2^{u} \frac{1}{t^3}\,dt$

Let $u = x^2$

$\frac{du}{dx} = 2x$

$$F'(x) = \frac{dF}{dx} = \frac{dF}{du}\cdot\frac{du}{dx} = \frac{d}{du}\left[F(x)\right]\cdot\frac{du}{dx}$$

$$= \frac{d}{du}\left[\int_2^u \frac{1}{t^3}\,dt\right]\cdot\frac{du}{dx}$$

$$= \frac{1}{u^3}\cdot\frac{du}{dx} = \frac{1}{(x^2)^3}\cdot 2x = \frac{2x}{x^6} = \boxed{\frac{2}{x^5}}$$

92. $F(x) = \int_0^{x^2} \sin\theta^2\,d\theta = \int_0^{u} \sin\theta^2\,d\theta$

$u = x^2$

$\frac{du}{dx} = 2x$

$$F'(x) = \sin u^2\cdot\frac{du}{dx}$$

$$= \sin(x^2)^2\cdot 2x$$

$$= \boxed{2x\sin(x^4)}$$

88. $F(x) = \int_{-x}^{x} t^3\,dt = \int_{-x}^{a} t^3\,dt + \int_{a}^{x} t^3\,dt$

$= \underline{\int_a^x t^3\,dt} - \int_a^{-x} t^3\,dt$

$= x^3 - \left[(-x)^3 \cdot (-1)\right]$

$= \boxed{0}$

$$\boxed{\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f(g(x)) \cdot g'(x)}$$

## 4.5 Integration by Substitution

12. $\int x^2 (x^3+5)^4\,dx = \int (x^3+5)^4 \cdot x^2\,dx$

Let $u = x^3 + 5$

$\frac{du}{3} = \frac{\cancel{3}x^2\,dx}{\cancel{3}}$

$= \int u^4 \left(\frac{1}{3}\,du\right)$

$= \frac{1}{15}u^5 + C$

$= \boxed{\frac{1}{15}(x^3+5)^5 + C}$

22. $\int \frac{x^2}{(16-x^3)^2}\,dx = \int \frac{-du}{3u^2}$

$u = 16 - x^3$

$= \int -\frac{1}{3}u^{-2}\,du$

$\frac{du}{-3} = \frac{\cancel{-3}x^2dx}{\cancel{-3}}$

$= \frac{1}{3}u^{-1} + C$

$= \boxed{\frac{1}{3(16-x^3)} + C}$

4.4: 45-51 odd

HW 75-91 odd