24. $\int \sec^6 3x\,dx$

$= \int (1+\tan^2 3x)(1+\tan^2 3x)\sec^2 3x\,dx$

$= \int (1+2\tan^2 3x+\tan^4 3x)\sec^2 3x\,dx$

$= \int \sec^2 3x\,dx + \int 2\tan^2 3x\sec^2 3x\,dx + \int \tan^4 3x\sec^2 3x\,dx$

$\frac{1}{3}\tan 3x$

$u=\tan 3x$

$du=\frac{1}{3}\sec^2 3x\,dx$

$6du=2\sec^2 3x\,dx$

$\int 6u^2\,du=2u^3$

$2\tan^3 3x$

$u=\tan 3x$

$3du=\frac{1}{3}\sec^2 3x\,dx$

$\int 3u^4\,du=\frac{3}{5}u^5$

$\frac{3}{5}\tan^5 3x$

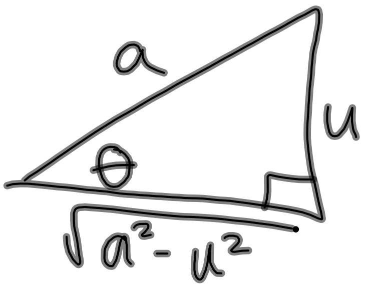
$$= \boxed{\frac{1}{3}\tan 3x + 2\tan^3 3x + \frac{3}{5}\tan^5 3x + C}$$

7.4 Trig Substitution

$\sqrt{a^2-u^2} = a\cos\theta$

$u = a\sin\theta$

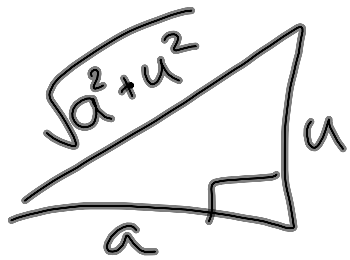
$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$



$\sqrt{a^2+u^2} = a\sec\theta$

$u = a\tan\theta$

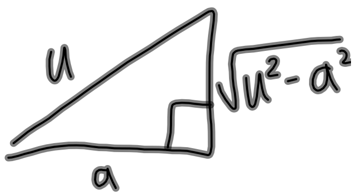
$, \; -\frac{\pi}{2} < \theta < \frac{\pi}{2}$



$\sqrt{u^2-a^2} = \begin{cases} +a\tan\theta, & u>a \\ -a\tan\theta, & u<-a \end{cases}$

$u = a\sec\theta$

$0 \le \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \le \pi$



6. $\displaystyle\int \frac{10}{x^2\sqrt{25-x^2}}\,dx$

$x = 5\sin\theta$ $\qquad \sqrt{25-x^2} = \sqrt{25-(5\sin\theta)^2} =$

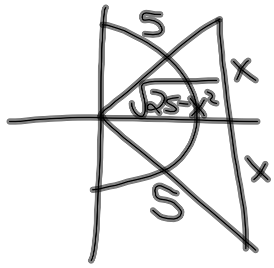
$dx = 5\cos\theta\, d\theta$ $\qquad = \sqrt{25(1-\sin^2\theta)} = 5\sqrt{\cos^2\theta}$

$\sin\theta = \frac{x}{5}$ $\qquad = 5\cos\theta$

$\theta = \sin^{-1}\frac{x}{5}$

$$\int \frac{10\,dx}{x^2\sqrt{25-x^2}} = \int \frac{10\cdot \cancel{5\cos\theta}\,d\theta}{(5\sin\theta)^2\cdot\cancel{5\cos\theta}} = \int \frac{2}{5}\csc^2\theta\,d\theta$$

$$= -\frac{2}{5}\cot\theta + C$$



$$= \boxed{-\frac{2}{5}\cdot\frac{\sqrt{25-x^2}}{x} + C}$$

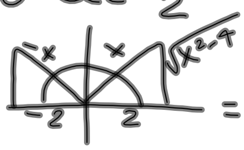
12. $\displaystyle\int \frac{x^3\,dx}{\sqrt{x^2-4}} = \int \frac{(2\sec\theta)^3\cdot 2\sec\theta\tan\theta\,d\theta}{\sqrt{(2\sec\theta)^2-4}}$

$x = 2\sec\theta$

$dx = 2\sec\theta\tan\theta\,d\theta$ $\qquad = \displaystyle\int \frac{16\sec^4\theta\tan\theta\,d\theta}{\sqrt{4(\sec^2\theta-1)}}$

$\sec\theta = \frac{x}{2}$

$\theta = \sec^{-1}\frac{x}{2}$ $\qquad = \displaystyle\int \frac{16\sec^4\theta\tan\theta\,d\theta}{2\sqrt{\tan^2\theta}} =$



$$= \int 8\sec^4\theta\,d\theta = 8\int(1-\tan^2\theta)\sec^2\theta\,d\theta$$

$$= 8\int\sec^2\theta\,d\theta - 8\int\tan^2\theta\sec^2\theta\,d\theta \qquad \int u^2\,du = \frac{1}{3}u^3$$

$$= 8\tan\theta - \frac{8}{3}\tan^3\theta \qquad u = \tan\theta,\ du = \sec^2\theta\,d\theta$$

$$= 8\cdot\frac{\sqrt{x^2-4}}{2} - \frac{8}{3}\left(\frac{\sqrt{x^2-4}}{2}\right)^3 + C$$

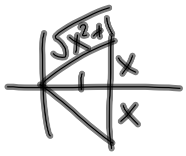
$$= \boxed{4\sqrt{x^2-4} - \frac{1}{3}(x^2-4)^{3/2} + C}$$

16. $$\int \frac{x^2\,dx}{(1+x^2)^2} = \int \frac{(\tan^2\theta)\cdot \sec^2\theta\, d\theta}{(1+\tan^2\theta)^2}$$

$x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

$\theta = \tan^{-1}\frac{x}{1}$



$$= \int \frac{\tan^2\theta\,\cancel{\sec^2\theta}\, d\theta}{(\sec^2\theta)^{\cancel{2}}}$$

$$= \int \frac{\sin^2\theta}{\cancel{\cos^2\theta}}\cdot\frac{\cancel{\cos^2\theta}}{1}\, d\theta = \int \sin^2\theta\, d\theta$$

$$= \int \frac{1-\cos 2\theta}{2}\, d\theta = \int\left(\frac{1}{2}-\frac{1}{2}\cos 2\theta\right)d\theta =$$

$$= \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C$$

$$= \frac{1}{2}\arctan x - \frac{1}{4}\left(2\sin\theta\cos\theta\right) + C$$

$$= \frac{1}{2}\arctan x - \frac{1}{2}\cdot\frac{x}{\sqrt{x^2+1}}\cdot\frac{1}{\sqrt{x^2+1}} + C$$

$$= \boxed{\frac{1}{2}\arctan x - \frac{x}{2(x^2+1)} + C}$$

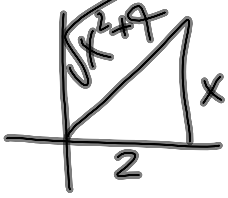
30. $$\int \frac{dx}{x\sqrt{4x^2+16}} = \int \frac{\cancel{2}\sec^2\theta\, d\theta}{\cancel{2}\tan\theta\sqrt{16(\tan^2 x+1)}} =$$

$x = 2\tan\theta$

$dx = 2\sec^2\theta\, d\theta$

$\tan\theta = \frac{x}{2}$

$\theta = \tan^{-1}\frac{x}{2}$



$$= \int \frac{\sec^{\cancel{2}}\theta\, d\theta}{\tan\theta\cdot 4\,\cancel{\sec\theta}}$$

$$= \frac{1}{4}\int \frac{\frac{1}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}}\, d\theta = \frac{1}{4}\int \frac{1}{\cancel{\cos\theta}}\cdot\frac{\cancel{\cos\theta}}{\sin\theta}\, d\theta$$

$$= \frac{1}{4}\int \csc\theta\, d\theta = \frac{1}{4}\left[-\ln\left|\csc\theta+\cot\theta\right| + C\right]$$

$$= -\frac{1}{4}\ln\left|\frac{\sqrt{x^2+4}}{x}+\frac{2}{x}\right| + C$$

40. $\int x \arcsin x \, dx = \frac{1}{2}x^2 \arcsin x - \int \frac{x^2 dx}{2\sqrt{1-x^2}}$

$u = \arcsin x$ $dv = x\,dx$ subst. $x = \sin\theta$

$du = \frac{dx}{\sqrt{1-x^2}}$ $v = \frac{1}{2}x^2$

Subst $x = \sin\theta$

$\frac{1}{2}x^2 \arcsin x - \frac{1}{4}\arcsin x + \frac{1}{4}x\sqrt{1-x^2} + C$

Subs. $x = \cos\theta$

$\frac{1}{2}x^2 \arcsin x + \frac{1}{4}\arccos x + \frac{1}{4}x\sqrt{1-x^2} + C$