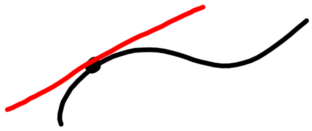
3.9 - Differentials

$(c, f(c)) = (x_1, y_1)$



Recall:

For a function f that is differentiable at c, the equation of the tangent line at the point $(c, f(c))$ is given by

$$y - f(c) = f'(c)(x - c)$$

This follows from the point-slope equation $y - y_1 = m(x - x_1)$, where the slope m is the derivative $f'(x)$ evaluated at the point $(c, f(c))$.

Since c, $f(c)$, and $f'(c)$ are all constants, if we rearrange to solve for y,

$$y = f(c) + f'(c)(x - c)$$

y is a linear function of x, called the linear approximation or tangent line approximation to the graph of $f(x)$ at $x = c$.

$$T(x) = f(c) + f'(c)(x - c)$$

For values of x close to c, values of $y = T(x)$ can be used as approximations of the values of the original function f.

Recall that the slope of the *secant line* through two points $(c, f(c))$ and $(x, f(x))$ is given by $\frac{\Delta y}{\Delta x} = \frac{f(x) - f(c)}{x - c}$, and the slope of the *tangent line* is the limit as the distance between these two points goes to zero of this expression, which we define to be the derivative.

Noting that the change in x is $\Delta x = x - c$, or $x = c + \Delta x$ and hence $f(x) = f(c + \Delta x)$, we can write this two ways:

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x}$$

Actual change in y is $\Delta y = f(x) - f(c) = f(c + \Delta x) - f(c)$.

Recalling the tangent line *approximation* equation

$$T(x) = f(c) + f'(c)(x - c) = f(c) + f'(c)\Delta x$$

We can see that change in y can be approximated by $T(x) - f(c)$, or

Approximate change in y is $\Delta y \approx f'(c)\Delta x$.

For such an approximation, Δx is denoted dx, and is called the differential of x. The expression $f'(x)dx$ is denoted by dy and called the **differential of y**.

$$\boldsymbol{dy = f'(x)dx} \qquad dy = y'dx$$

In many applications, the differential of y can be used as an approximation of the actual change in y, i.e. $\Delta y \approx f'(x)dx$

All of the differentiation rules can be written in differential form.

By definition of differentials, we have for functions (of x) u and v:

$du = u'dx$ and $dv = v'dx$

Note that rearranged, these look like $\frac{du}{dx} = u'$ and $\frac{dv}{dx} = v'$.

For example, the Product Rule becomes:

$d[uv] = [uv]'dx = [uv' + vu']dx = uv'dx + vu'dx = udv + vdu$

Differential Formulas

Constant multiple: $d[cu] = cdu$

Sum or difference: $d[u \pm v] = du \pm dv$

Product: $d[uv] = udv + vdu$

Quotient: $d\left[\frac{u}{v}\right] = \frac{vdu - udv}{v^2}$

3.9 #2 $f(x) = \frac{6}{x^2}$; $\left(2, \frac{3}{2}\right) \leftarrow (c, f(c))$

Compare the actual function values with the tangent line approximation near 2.

$f(x) = 6x^{-2}$

$f'(x) = -12x^{-3} = \frac{-12}{x^3}$

$f'(2) = \frac{-12}{2^3} = -\frac{12}{8} = -3/2$

Tangent line $T(x)$: $y = f(c) + f'(c)(x - c)$

$T(x) = \frac{3}{2} + \frac{-3}{2}(x - 2) = \frac{3}{2} - \frac{3}{2}x + 3$

$T(x) = \frac{9}{2} - \frac{3}{2}x$

$f(x) = \frac{6}{x^2}$

x	1.9	1.99	2	2.01	2.1
$f(x)$	1.6620	1.5151	1.5	1.4851	1.3605
$T(x)$	1.6500	1.5150	1.5	1.4850	1.3500

3.9 #8 $y = 1 - 2x^2 = f(x)$; $x = 0$ $c=0$; $\Delta x = dx = -0.1$

Compare dy and Δy for the given values of x and Δx. $f'(x) = -4x$

$\boldsymbol{\Delta y = f(c + \Delta x) - f(c)}$

$= 1 - 2(0 + (-0.1))^2 - (1 - 2(0)^2)$

$= 1 - 2(0.1)^2 - 1$

$= 1 - 0.02 - 1$

$= -0.02$

$\boldsymbol{dy = f'(x)dx}$ $f'(c)dx$

$= -4(0)\cdot(-0.1)$

$= 0$

Find the differential dy.

$dy = f'(x)dx$

12. $y = 3x^{2/3}$

$dy = 2x^{-1/3}dx$

16. $y = \sqrt{x} + \frac{1}{\sqrt{x}} = x^{1/2} + x^{-1/2}$

$dy = \frac{1}{2}x^{-1/2}dx - \frac{1}{2}x^{-3/2}dx$

$= \frac{dx}{2\sqrt{x}} - \frac{dx}{2\sqrt{x^3}}$

14. $y = \sqrt{9 - x^2} = (9 - x^2)^{1/2}$

$dy = \frac{1}{2}(9 - x^2)^{-1/2}\cdot(-2x)dx$

20. $y = \frac{\sec^2 x}{x^2 + 1} = \frac{(\sec x)^2}{x^2 + 1}$

$dy = 2\sec x\cdot\sec x\tan x$

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$$y = \frac{\sec^2 x}{x^2+1} = \frac{(\sec x)^2}{x^2+1}$$

$$dy = \frac{2\sec x \cdot \sec x \tan x \cdot (x^2+1) - \sec^2 x \cdot 2x}{(x^2+1)^2}\,dx$$

3.9 #46

Use differentials to approximate $\sqrt[3]{26}$

calculator says $26^{1/3} \approx 2.962496068\ldots$

$$\left.\begin{aligned} \Delta y &= f(c+\Delta x) - f(c) \\ dy &= f'(x)dx \\ \Delta y &\approx dy \end{aligned}\right\} \rightarrow \boldsymbol{f(c+\Delta x) - f(c) \approx f'(x)dx}$$

$$\boxed{f(c+\Delta x) \approx f(c) + f'(c)dx}$$

$f(x) = \sqrt[3]{x} = x^{1/3}$; $c = 27$; $\Delta x = dx = -1$

$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$

$\sqrt[3]{26} = \sqrt[3]{27+(-1)} \approx \sqrt[3]{27} + \frac{1}{3\sqrt[3]{27^2}} \cdot (-1)$

$= 3 + \frac{1}{27}(-1) = 3 - \frac{1}{27} = \frac{80}{27}$

$\approx 2.9\overline{62}$

Recall rules of exponents: $x^{m/n} = (x^m)^{1/n} = \left(x^{1/n}\right)^m$
$= \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m$

3.9 #50

Use differentials to approximate $\tan(0.05)$.

$f(c + \Delta x) \approx f(c) + f'(x)dx$

$f(x) =$; $c =$; $\Delta x = dx =$

Homework:

3.9 #5, 9; 11-19 odd; 45, 49