16. $\int x^2 \sin^2 x\, dx$

$\cos 2x = 2\cos^2 x - 1$

$\cos 2x = 1 - 2\sin^2 x$

$\sin^2 x = \dfrac{1-\cos 2x}{2}$

$u = x^2 \quad dv = \sin^2 x\, dx$

$du = 2x\,dx \quad v = \int \sin^2 x\, dx$

$= \int \frac{1}{2}dx - \int \frac{1}{2}\cos 2x\, dx$

$= \frac{1}{2}x - \frac{1}{4}\sin 2x$

$$\int x^2 \sin^2 x\, dx =$$

$$= x^2\left(\tfrac{1}{2}x - \tfrac{1}{4}\sin 2x\right) - \int \left(\tfrac{1}{2}x - \tfrac{1}{4}\sin 2x\right)\cdot 2x\,dx$$

$$= \tfrac{1}{2}x^3 - \tfrac{1}{4}x^2\sin 2x - \underbrace{\int x^2 dx}_{\frac{1}{3}x^3} + \int \tfrac{1}{2}x\sin 2x\,dx$$

$$= \tfrac{1}{6}x^3 - \tfrac{1}{4}x^2 \sin 2x + \int \tfrac{1}{2}x\sin 2x\,dx$$

$u = \frac{1}{2}x \quad dv = \sin 2x\,dx$

$du = \frac{1}{2}dx \quad v = -\frac{1}{2}\cos 2x$

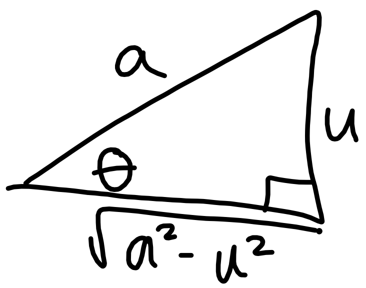
$$= \tfrac{1}{6}x^3 - \tfrac{1}{4}x^2\sin 2x - \tfrac{1}{4}x\cos 2x - \int -\tfrac{1}{4}\cos 2x\,dx$$

$$= \boxed{\tfrac{1}{6}x^3 - \tfrac{1}{4}x^2\sin 2x - \tfrac{1}{4}x\cos 2x + \tfrac{1}{8}\sin 2x + C}$$

7.4 Trig Substitution

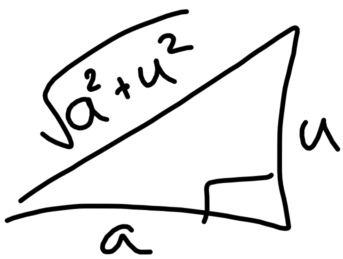
$\sqrt{a^2-u^2} = a\cos\theta \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$u = a\sin\theta$



$\sqrt{a^2+u^2} = a\sec\theta \quad , \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

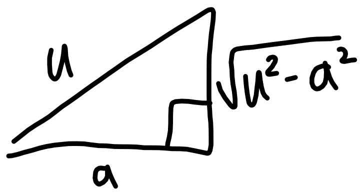
$u = a\tan\theta$



$$\sqrt{u^2-a^2} = \begin{cases} a\tan\theta, & u > a \\ -a\tan\theta, & u < -a \end{cases}$$

$u = a\sec\theta$

$0 \le \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \le \pi$



6. $\displaystyle\int \frac{10}{x^2\sqrt{25-x^2}}\,dx = \int \frac{10\cdot 5\cos\theta\, d\theta}{25\sin^2\theta\sqrt{25-25\sin^2\theta}}$

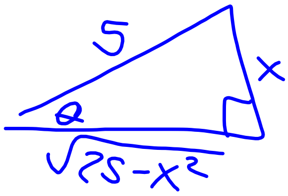
$x = 5\sin\theta \Rightarrow \sin\theta = \frac{x}{5}$

$\theta = \arcsin\left(\frac{x}{5}\right)$

$dx = 5\cos\theta\, d\theta$

$= \displaystyle\int \frac{2\cos\theta\, d\theta}{\sin^2\theta \cdot 5\,\underbrace{\sqrt{1-\sin^2\theta}}_{\cos^2\theta}}$

$= \displaystyle\int \frac{2\cancel{\cos\theta}\, d\theta}{5\sin^2\theta\cancel{\cos\theta}} = \int \frac{2}{5}\csc^2\theta\, d\theta$



$= -\frac{2}{5}\cot\theta + C$

$= -\frac{2}{5}\cot\left(\arcsin\frac{x}{5}\right) + C$

$= \boxed{-\frac{2}{5}\cdot\frac{\sqrt{25-x^2}}{x} + C}$

12. $\displaystyle\int \frac{x^3\,dx}{\sqrt{x^2-4}} = \int \frac{8\sec^3\theta \cdot 2\sec\theta\tan\theta\, d\theta}{\sqrt{(2\sec\theta)^2-4}}$

$\sqrt{4(\underbrace{\sec^2\theta-1}_{})}$

$\sqrt{4\tan^2\theta}$

$x = 2\sec\theta$

$dx = 2\sec\theta\tan\theta\, d\theta$

$= \displaystyle\int \frac{\overset{8}{\cancel{16}}\sec^4\theta\,\cancel{\tan\theta}\, d\theta}{\cancel{2\tan\theta}}$

$= \displaystyle\int 8\sec^4\theta\, d\theta = \int 8(\tan^2\theta+1)\sec^2\theta\, d\theta$

$u = \tan\theta$

$du = \sec^2\theta\, d\theta$

$= \displaystyle\int 8(u^2+1)\, du$

$= \frac{8}{3}u^3 + 8u + C = \frac{8}{3}\tan^3\theta + 8\tan\theta + C$

$\sec\theta = \frac{x}{2}$

$= \boxed{\frac{8}{3}\left(\frac{\sqrt{x^2-4}}{2}\right)^3 + 8\left(\frac{\sqrt{x^2-4}}{2}\right) + C}$

16. $\int \frac{x^2\,dx}{(1+x^2)^2} = \int \frac{\tan^2\theta \cdot \sec^2\theta\, d\theta}{(1+\tan^2\theta)^2}$

$x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

$= \int \frac{\tan^2\theta \cdot \sec^2\theta\, d\theta}{(\sec^2\theta)^2}$

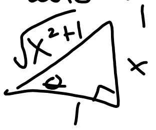
$= \int \frac{\tan^2\theta\, d\theta}{\sec^2\theta}$

$\cos 2\theta = 1 - 2\sin^2\theta$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$= \int \frac{\sin^2\theta}{\cancel{\cos^2\theta}} \cdot \cancel{\cos^2\theta}\, d\theta$

$\tan\theta = \frac{x}{1}$



$\theta = \arctan x$

$= \tan^{-1} x$

$= \int \sin^2\theta\, d\theta$

$= \int \frac{1-\cos 2\theta}{2}\, d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C$

$= \frac{1}{2}\arctan x - \frac{1}{4}(2\sin\theta\cos\theta) + C$

$= \frac{1}{2}\arctan x - \frac{1}{2} \cdot \frac{x}{\sqrt{x^2+1}} \cdot \frac{1}{\sqrt{x^2+1}} + C$

$= \boxed{\frac{1}{2}\arctan x - \frac{x}{2x^2+2} + C}$

Homework:

- 6.1 #1-9 odd; 19, 43 (area between curves)
- 6.2 #11, 13, 17, 19, 21, 25, 29, 35 (volume of solid of revolution)
- 6.4 #5, 7, 13, 33, 35 (arc length and surface of revolution)

- 7.1 #5-53 odd (basic integration rules)
- 7.2 #1-35odd (integration by parts)

- 7.3 #3-15odd; 21-37odd; 47-67odd (trigonometric integrals)
- 7.4 #5-15odd; 19-43odd (trigonometric substitution)