Homework for Test #1

- 3.9 #5, 9; 11 19 odd; 45, 49
- 4.1 #5 33 odd; 55 61 odd; 67, 83
- 4.2 #7-19odd; 27-37odd; 41,43,47,53
- 4.3 #7,17,37,43,45
- 4.4 #13, 15, 23, 31

Hw: #20-26 on THQ

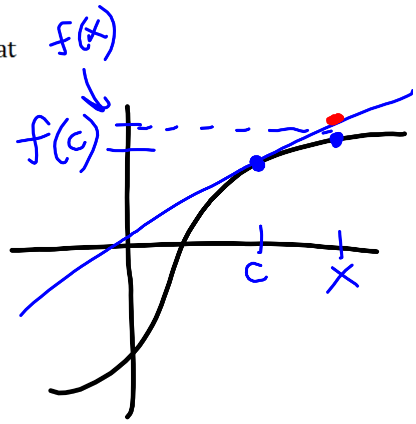
HW 4.4: 45-51 odd
75-91 odd

4.5 # 7-33, 41-53, 57-63 odd

3.9 - Differentials

Recall:
For a function f that is differentiable at c, the equation of the tangent line at the point $(c, f(c))$ is given by

$y - f(c) = f'(c)(x - c)$



This follows from the point-slope equation $y - y_1 = m(x - x_1)$, where the slope m is the derivative $f'(x)$ evaluated at the point $(c, f(c))$.

Since c, $f(c)$, and $f'(c)$ are all constants, if we rearrange to solve for y,

$y = f(c) + f'(c)(x - c)$

y is a linear function of x, called the linear approximation or tangent line approximation to the graph of $f(x)$ at $x = c$.

$T(x) = f(c) + f'(c)(x - c)$

For values of x close to c, values of $y = T(x)$ can be used as approximations of the values of the original function f.

Recall that the slope of the *secant line* through two points $(c, f(c))$ and $(x, f(x))$ is given by $\frac{\Delta y}{\Delta x} = \frac{f(x)-f(c)}{x-c}$, and the slope of the *tangent line* is the limit as the distance between these two points goes to zero of this expression, which we define to be the derivative.

Noting that the change in x is $\Delta x = x - c$, or $x = c + \Delta x$ and hence $f(x) = f(c + \Delta x)$, we can write this two ways:

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x}$$

Actual change in y is $\Delta y = f(x) - f(c) = f(c + \Delta x) - f(c)$.

Recalling the tangent line *approximation* equation

$T(x) = f(c) + f'(c)(x - c) = f(c) + f'(c)\Delta x$

We can see that change in y can be approximated by $T(x) - f(c)$, or

Approximate change in y is $\Delta y \approx f'(c)\Delta x$.

For such an approximation, Δx is denoted dx, and is called the differential of x. The expression $f'(x)dx$ is denoted by dy and called the **differential of y**.

$$\boldsymbol{dy = f'(x)dx}$$

In many applications, the differential of y can be used as an approximation of the actual change in y, i.e. $\Delta y \approx f'(x)dx$

All of the differentiation rules can be written in differential form.

By definition of differentials, we have for functions (of x) u and v:

$du = u'dx$ and $dv = v'dx$

$du = \frac{du}{dx} \cdot dx$

Note that rearranged, these look like $\frac{du}{dx} = u'$ and $\frac{dv}{dx} = v'$.

For example, the Product Rule becomes:

$d[uv] = [uv]'dx = [uv' + vu']dx = uv'dx + vu'dx = udv + vdu$

Differential Formulas

Constant multiple:	$d[cu] = cdu$
Sum or difference:	$d[u \pm v] = du \pm dv$
Product:	$d[uv] = udv + vdu$
Quotient:	$d\left[\frac{u}{v}\right] = \frac{vdu - udv}{v^2}$

Find the differential dy.

$dy = f'(x)dx$

12. $y = 3x^{2/3}$

$dy = 2x^{-1/3}dx$

16. $y = \sqrt{x} + \frac{1}{\sqrt{x}} = x^{1/2} + x^{-1/2}$

$dy = \left(\frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2}\right)dx$

$= \frac{dx}{2\sqrt{x}} - \frac{dx}{2x^{3/2}}$

14. $y = \sqrt{9 - x^2}$

$dy = \frac{1}{2}(9-x^2)^{-1/2}(-2x)\,dx$

20. $y = \frac{\sec^2 x}{x^2 + 1}$

$dy = \frac{(x^2+1)(2\sec x \cdot \sec x \tan x) - 2x\sec^2 x}{(x^2+1)^2}\,dx$

<u>3.9 #2</u> $f(x) = \frac{6}{x^2}$; $\left(2, \frac{3}{2}\right)$

Compare the actual function values with the tangent line approximation near 2.

$f(x) = 6x^{-2}$

$f'(x) = -12x^{-3} = \frac{-12}{x^3}$

$f'(2) = \frac{-12}{8} = -\frac{3}{2}$

$c = 2$

$f(c) = f(2) = \frac{3}{2}$

Tangent line $T(x)$: $y = f(c) + f'(c)(x - c)$

$T(x) = \frac{3}{2} + \left(\frac{-3}{2}\right)(x-2)$

x	1.9	1.99	2	2.01	2.1
$f(x)$	1.66205	1.515113	3/2	1.48511	1.360544
$T(x)$	1.65	1.515	3/2	1.485	1.35

3.9 #8 $y = 1 - 2x^2 = f(x)$; $x = 0$; $\Delta x = dx = -0.1$ $\quad f'(x) = -4x$

Compare dy and Δy for the given values of x and Δx.

$\boldsymbol{\Delta y = f(c + \Delta x) - f(c)}$ $\qquad$ $\boldsymbol{dy = f'(c)dx}$

$$\Delta y = 1 - 2(0 + (-0.1))^2 - (1 - 2(0)^2)$$
$$= 1 - 2(0.01) - 1 = \boxed{-0.02}$$
$$dy = -4(0)\cdot(-0.1) = \boxed{0}$$

3.9 #46

Use differentials to approximate $\sqrt[3]{26}$

calculator:

$\sqrt[3]{26} = 2.962496$

$\frac{80}{27} = 2.96296$

$$\left.\begin{aligned} \Delta y &= f(c + \Delta x) - f(c) \\ dy &= f'(x)dx \\ \Delta y &\approx dy \end{aligned}\right\} \to \boldsymbol{f(c + \Delta x) - f(c) \approx f'(x)dx}$$

$$\boxed{f(c + \Delta x) \approx f(c) + f'(c)dx}$$

$f(x) = \sqrt[3]{x} = x^{1/3}$; $c = 27$; $\Delta x = dx = -1$

$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3(\sqrt[3]{x})^2}$ $\quad f'(c) = \frac{1}{3(\sqrt[3]{27})^2} = \frac{1}{27}$

$\sqrt[3]{26} = \sqrt[3]{27 + (-1)} \approx$

$$\approx \sqrt[3]{27} + \frac{1}{27}\cdot(-1) = 3 - \frac{1}{27} = \boxed{\frac{80}{27}}$$

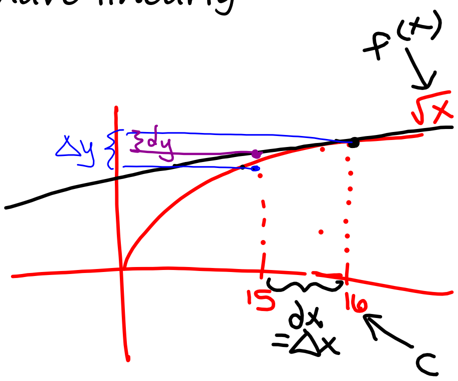
Recall rules of exponents: $x^{m/n} = (x^m)^{1/n} = \left(x^{1/n}\right)^m$
$= \sqrt[n]{x^m} \quad = \left(\sqrt[n]{x}\right)^m$

Why does the differential give us a good approximation for the actual change in y?

locally, functions behave linearly

$\sqrt{15} = \sqrt{16-1}$

$\approx \sqrt{16} + \frac{1}{2\sqrt{16}} \cdot (-1)$



$y - y_1 = m(x - x_1)$

$y \approx \underbrace{y_1}_{f(c)} + \underbrace{m}_{f'(c)} \underbrace{(x - x_1)}_{\Delta x}$

3.9 #50

$\tan(0) = 0$

Use differentials to approximate $\tan(0.05)$.

$f(c + \Delta x) \approx f(c) + f'(c)dx$

$f(x) = \tan x$; $c = 0$; $\Delta x = dx = 0.05$

$\tan(0.05) = \tan(0 + 0.05)$

$\approx \tan(0) + \sec^2(0) \cdot 0.05$

$= 0 + 1 \cdot 0.05 = \boxed{0.05}$