REIDEMEISTER MOVES

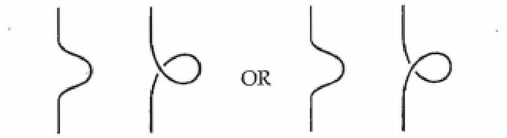


Figure 1.22 Type I Reidemeister move.

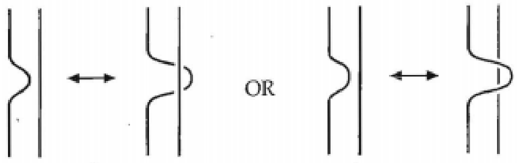


Figure 1.23 Type II Reidemeister move.

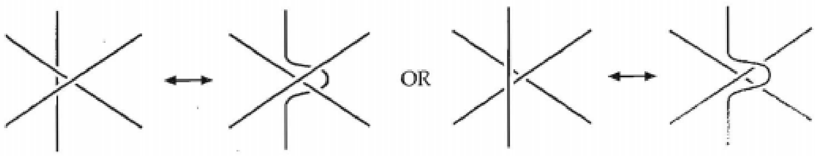
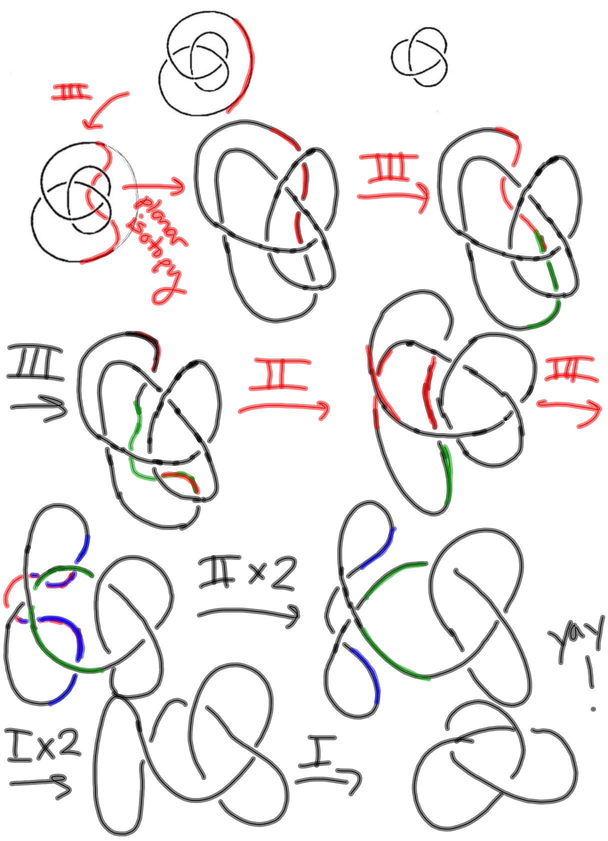


Figure 1.24 Type III Reidemeister move.

Exercise 1.10 Show that the two projections in Figure 1.28 represent the same knot by finding a series of Reidemeister moves from one to the other.



LINKS

A **link** is a set of knotted loops all tangled up together. Two links are considered to be the same if we can deform the one link to the other link without ever having any one of the loops intersect itself or any of the other loops in the process. Here are two projections of one of the simplest links, known as the **Whitehead link** (Figure 1.30).

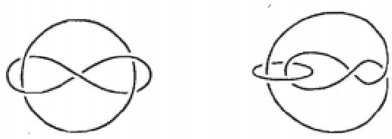
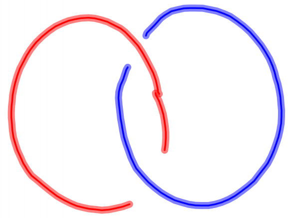


Figure 1.30 Two projections of the Whitehead link.



Exercise 1.13 Show that the two projections represent the same link.

Since it is made up of two loops knotted with each other, we say that it is a **link of two components**. Here is another well-known link with three components, called the **Borromean rings** (Figure 1.31). This link is named after the Borromeas, an Italian family from the Renaissance that used this pattern of interlocking rings on their family crest.

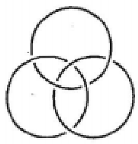
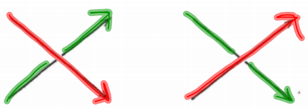


Figure 1.31 The Borromean rings.

LINKING NUMBER

Let M and N be two components in a link, and choose an orientation on each of them. Then at each crossing between the two components, one of the pictures in Figure 1.34 will hold. We count a $+1$ for each crossing of the first type, and a -1 for each crossing of the second type. Sometimes it is hard to determine from the picture whether a crossing is of the first type or the second type. Note that if a crossing is of the first type, then rotating the understrand clockwise lines it up with the overstrand so that their arrows match. Similarly, if a crossing is of the second type, then rotating the understrand counterclockwise lines the understrand up with the overstrand so that their arrows match.



+1 -1

Figure 1.34 Computing linking number.

Now, we will take the sum of the $+1$s and -1s over all the crossings between M and N and divide this sum by 2. This will be the linking number. We do not count the crossings between a component and itself. For the unlink, the linking number of the two components is 0. For the Hopf link, the linking number will be 1 or -1, depending on the orientations on the two components. The two components in the oriented link pictured in Figure 1.35 have linking number 2. Notice that if we reverse the orientation on one of the two components, but not the other, the linking number of these two components is multiplied by -1. If we just look at the absolute value of the linking number, however, it is independent of the orientations on the two components.

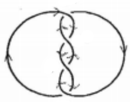
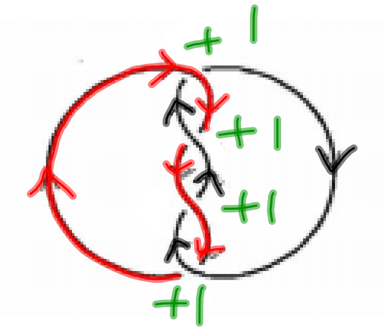


Figure 1.35 Linking number 2.

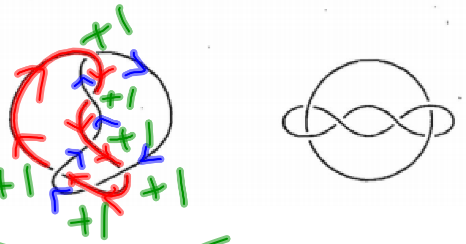


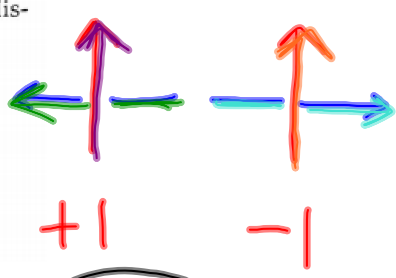
linking # :

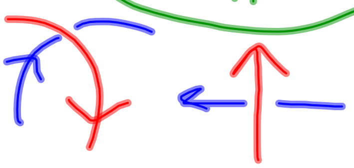
$$\frac{+1+1+1+1}{2} = \boxed{2}$$

Exercise 1.17 Compute the absolute values of the linking numbers of the two links shown in Figure 1.39 in order to show that they must be distinct links.

assigned exercise

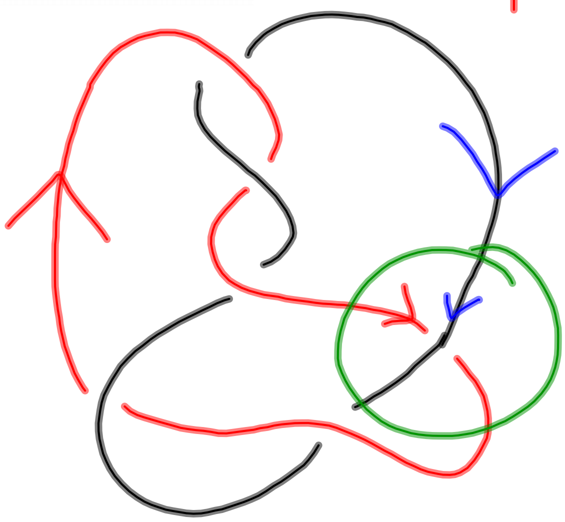






linking #

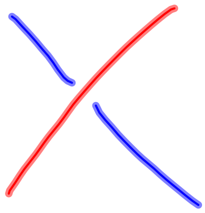
$$\frac{6}{2} = \boxed{3}$$



TRICOLORABILITY

colorability.

We will say that a **strand** in a projection of a link is a piece of the link that goes from one undercrossing to another with only overcrossings in between. We will say that a projection of a knot or link is **tricolorable** if each of the strands in the projection can be colored one of three different colors, so that at each crossing, either three different colors come together or all the same color comes together. In order that a projection be tricolorable, we further require that at least two of the colors are used. Figure 1.41 shows that these two projections of the trefoil knot are tricolorable (using white, gray, and black as the colors).



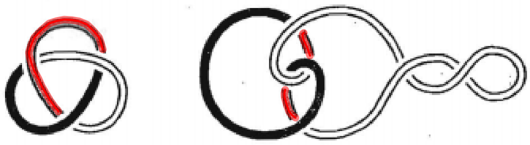
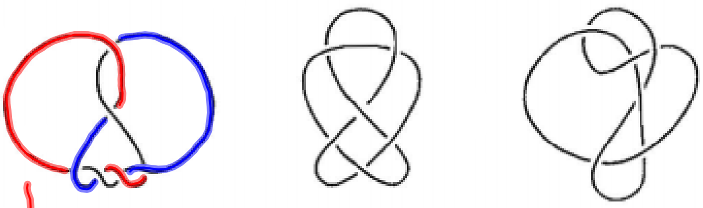


Figure 1.41 The trefoil is tricolorable.

In the first tricoloration, three different colors come together at each crossing, whereas in the second tricoloration, some of the crossings have only one color occurring. But none of the crossings in either picture have exactly two colors occurring, so these are valid tricolorations.

Exercise 1.21 Determine which of the projections of the three six-crossing knots 6_1, 6_2, and 6_3 in Figure 1.42 are tricolorable.



yes!

6_1 6_2 6_3

Figure 1.42 Projections of 6_1, 6_2, and 6_3.