<u>Updated/Revised deadlines</u>:

Thurs. 12/6 - Knot Theory due

Mon. 12/10 - Outline & Sources due

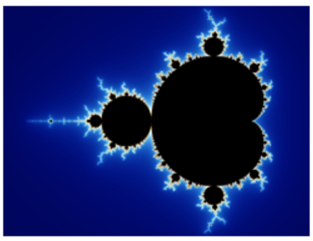
Thurs. 12/13 - Fractals due

FRACTALS

Fractals:

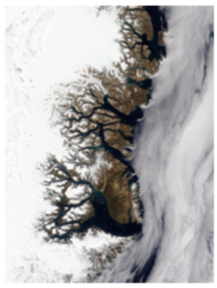
- are self-similar (at least approximately), i.e. have the rescaling property (when you zoom in on a piece it looks like the whole)
- have fine structure on arbitrarily small scales
- often have simple, recursive definitions

Mandelbrot Set

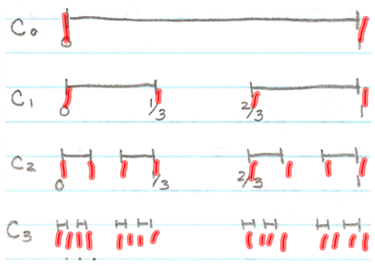


Romanesco broccoli

Coastline of Norway



The Cantor Middle-Third Set:



$1 = 2^0$ interval of length $1 = \left(\frac{1}{3}\right)^0$

$2 = 2^1$ intervals of length $\frac{1}{3} = \frac{1}{3}$

$4 = 2^2$ intervals of length $\frac{1}{9}$ $\left(\frac{1}{3}\right)^2$

$8 = 2^3$ intervals of length $\frac{1}{27} = \left(\frac{1}{3}\right)^3$

To obtain C_{n+1} from C_n, we remove the middle third of each interval in C_n. The Cantor set C is the intersection of all C_n. C is a fractal.

2^n intervals of length $\frac{1}{3^n}$

C_n consists of 2^n closed intervals of length $\frac{1}{3^n}$. The total length of C_n is $\left(\frac{2}{3}\right)^n$, which approaches 0 as n approaches ∞. Hence the "length" of C is 0.

Another way to state this is that the length of $C_{n+1} = \frac{2}{3} \cdot length\ of\ C_n$. Given this recursive definition, again we have that the length of C_n is , which approaches 0 as n approaches ∞.

The total length removed from the interval $[0,1]$ in the construction of the Cantor set is

$$\frac{1}{3}+\frac{2}{9}+\frac{4}{27}+\cdots=\sum_{n=0}^{\infty}\frac{2^n}{3^{n+1}}=\sum_{n=0}^{\infty}\frac{1}{3}\left(\frac{2}{3}\right)^n=\frac{\frac{1}{3}}{1-\frac{2}{3}}=\frac{\frac{1}{3}}{\frac{1}{3}}=\mathbf{1}$$

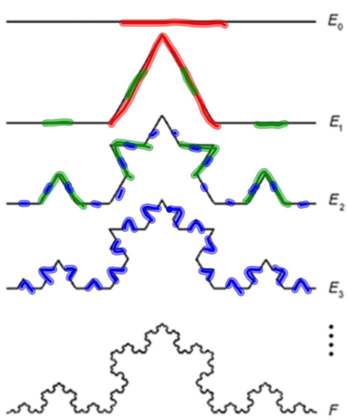
Hence we have a set from which its entire length has been removed. Yet there are still infinitely many points left in the set. Which points are they?

endpoints of the intervals

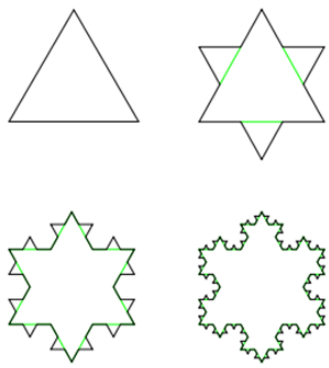
Note: the sum of an infinite geometric series with common ratio less than 1 in absolute value is equal to $S_\infty = \frac{a_1}{1-r}$, where a_1 is the first term and r is the common ratio.

$$\sum_{n=0}^{\infty} a^n = \frac{a_0}{1-r}$$

Koch Curve:



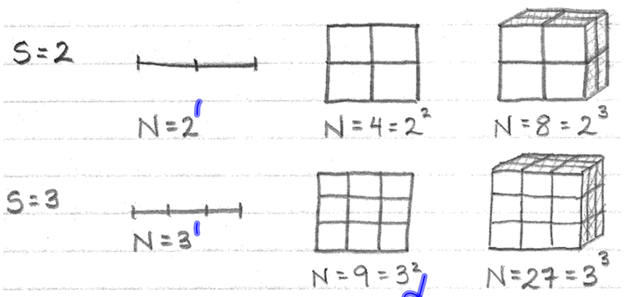
Koch Snowflake:

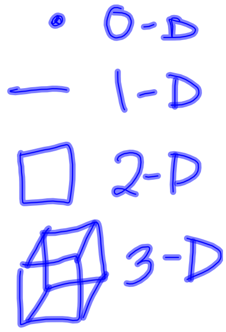


Fractal Dimension:

Recall that points in space are 0-dimensional; lines are 1-dimensional; a square is 2-dimensional; and a cube is 3-dimensional. Fractals don't behave exactly like objects in these integer dimensions.

Suppose that an object has the following property: if we scale it down by a factor of **S**, then the object can be built from **N** scaled-down versions of itself.





$N = S^d$, where $d =$ dimension

$d = \frac{\log N}{\log S} =$ fractal dimension

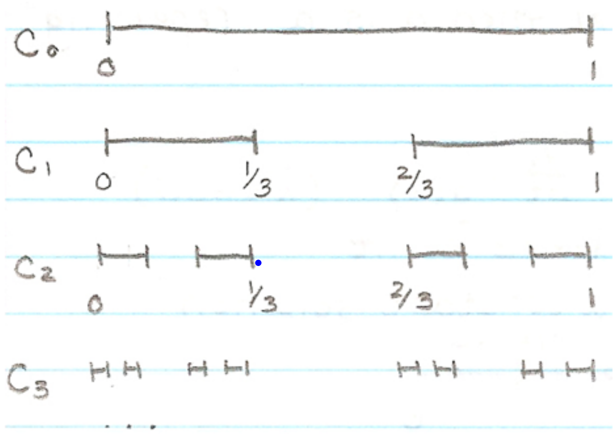
$$N = S^d$$

$$\log N = \log S^d$$

$$\log N = d \cdot \log S$$

$$d = \frac{\log N}{\log S}$$

The Cantor Middle-Third Set:

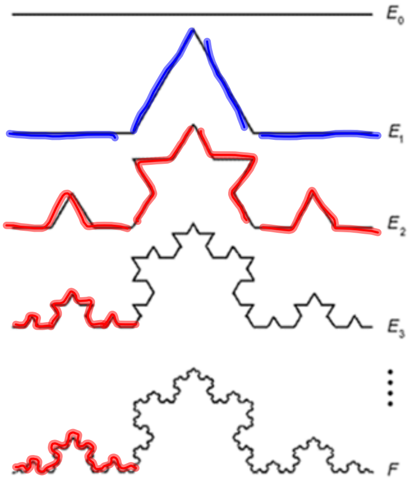


Suppose that an object has the following property: if we scale it down by a factor of **S**, then the object can be built from **N** scaled-down versions of itself.

$d = \frac{\log N}{\log S} =$ fractal dimension

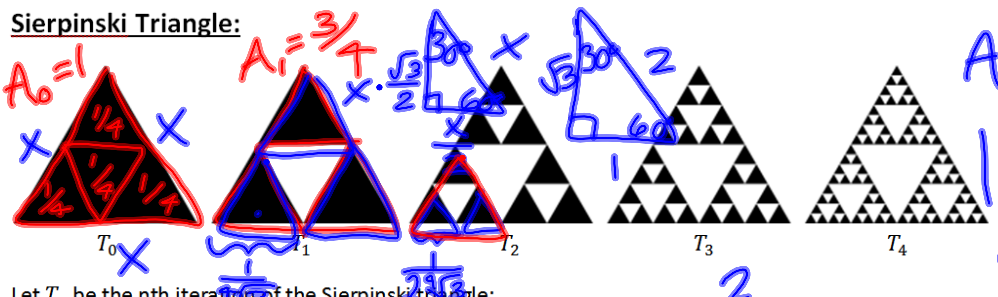
Ex Cantor Set $S = 3$, $N = 2$, $d = \frac{\log 2}{\log 3} \approx 0.63$

Koch Curve:



$$S = 3\ ,\ N = 4\ ,\ d = \frac{\log 4}{\log 3} \approx 1.62$$

Sierpinski Triangle:



$A = \frac{1}{2} bh$

$1 = \frac{1}{2} x \cdot x \cdot \frac{\sqrt{3}}{2}$

$1 = \frac{\sqrt{3}}{4} x^2$

$\frac{4}{\sqrt{3}} = x^2$

$\frac{2}{\sqrt[4]{3}} = x$

Let T_n be the nth iteration of the Sierpinski triangle;
let A_n be the area of the nth iteration of the Sierpinski triangle; let $A_0 = 1$;
let P_n be the perimeter of the nth iteration of the Sierpinski triangle.

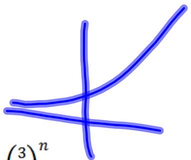
T_n consists of 3^n triangles of area $\left(\frac{1}{4}\right)^n$;

How much was removed?

We remove 3^n triangles of area $\left(\frac{1}{4}\right)^{n+1}$ to get T_{n+1} from T_n.

Hence the total area removed is:

$$\sum_{n=0}^{\infty} \frac{1}{4}\left(\frac{3}{4}\right)^n = \frac{\frac{1}{4}}{1-\frac{3}{4}} = \frac{\frac{1}{4}}{\frac{1}{4}} = 1$$

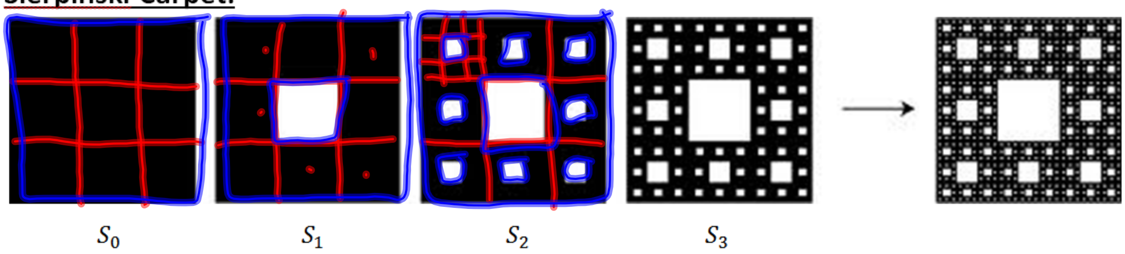


$P_{n+1} = \frac{3}{2} P_n$, which implies that $P_n = \left(\frac{3}{2}\right)^n$,
which approaches ∞ as n approaches ∞.
The Sierpinski triangle has zero area but an infinite perimeter!

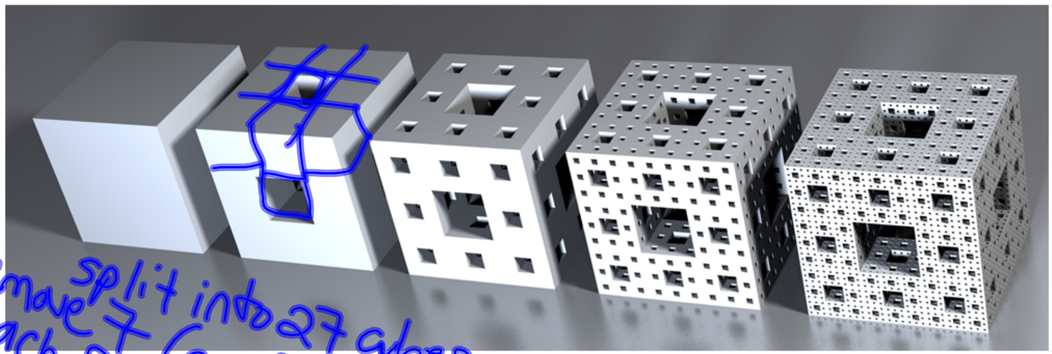
Fractal Dimension: $S = 2\ ,\ N = 3\ ,\ d = \frac{\log 3}{\log 2} \approx 1.58$

	T_0	T_1
Perimeter of this iteration	$P_0 = \frac{6}{\sqrt[4]{3}}$	$P_1 = \frac{9}{\sqrt[4]{3}}$
Perimeter of this iteration divided by perimeter of previous iteration	N/A	$\frac{P_1}{P_0} = \frac{3}{2}$
Area removed from previous iteration to obtain this iteration	N/A	1/4
Total area removed up to this point	0	1/4
Number of triangles in this iteration	1	3
Area of a triangle in this iteration	1	1/4
Total area of this iteration	$A_0 = 1$	$A_1 = 3/4$

Sierpinski Carpet:



Menger Sponge:



split into 27 cubes;
remove 7 (one from
each face & center)