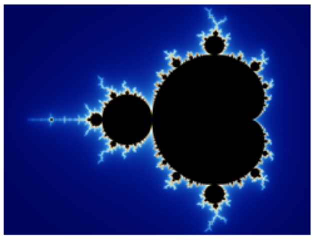
FRACTALS

Fractals:

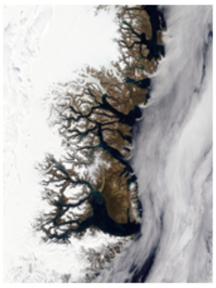
- are self-similar (at least approximately), i.e. have the rescaling property (when you zoom in on a piece it looks like the whole)
- have fine structure on arbitrarily small scales
- often have simple, recursive definitions

Mandelbrot Set

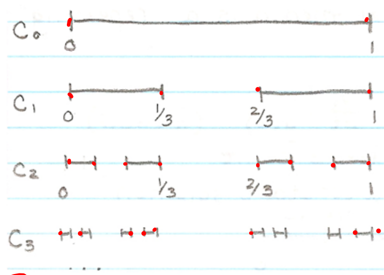


Romanesco broccoli

Coastline of Norway



The Cantor Middle-Third Set:



$2^0 = 1$ segment of length $1 = \frac{1}{3^0}$

$2^1 = 2$ segments of length $\frac{1}{3} = \frac{1}{3^1}$

$2^2 = 4$ segments of length $\frac{1}{9} = \frac{1}{3^2}$

$2^3 = 8$ segments of length $\frac{1}{27} = \frac{1}{3^3}$

C_n 2^n segments of length $\frac{1}{3^n}$

To obtain C_{n+1} from C_n, we remove the middle third of each interval in C_n. The Cantor set C is the intersection of all C_n. C is a fractal.

C_n consists of 2^n closed intervals of length $\frac{1}{3^n}$. The total length of C_n is $\left(\frac{2}{3}\right)^n$, which approaches 0 as n approaches ∞. Hence the "length" of C is 0.

Another way to state this is that the length of $C_{n+1} = \frac{2}{3} \cdot \textit{length of } C_n$. Given this recursive definition, again we have that the length of C_n is , which approaches 0 as n approaches ∞.

The total length <u>removed</u> from the interval $[0, 1]$ in the construction of the Cantor set is

$$\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \cdots = \sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^n = \frac{\frac{1}{3}}{1 - \frac{2}{3}} = \frac{\frac{1}{3}}{\frac{1}{3}} = \mathbf{1}$$

Hence we have a set from which its entire length has been removed. Yet there are still infinitely many points left in the set. <u>Which points are they</u>?

the endpoints of the intervals!

<u>Note</u>: the sum of an infinite geometric series with common ratio less than 1 in absolute value is equal to $S_\infty = \frac{a_1}{1-r}$, where a_1 is the first term and r is the common ratio.

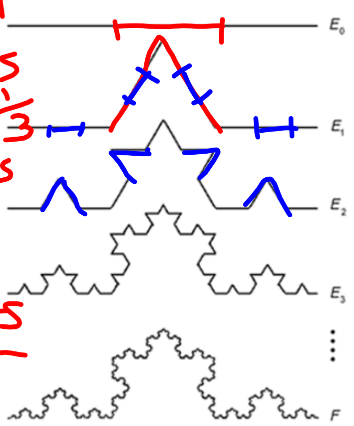
Koch Curve:

1 segment of length 1

4 segments of length $\frac{1}{3}$

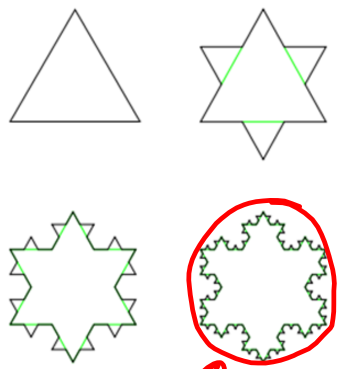
16 segments of length $1/9$

64 segments of length $1/27$



n: 4^n segments of length $\frac{1}{3^n}$

Koch Snowflake:

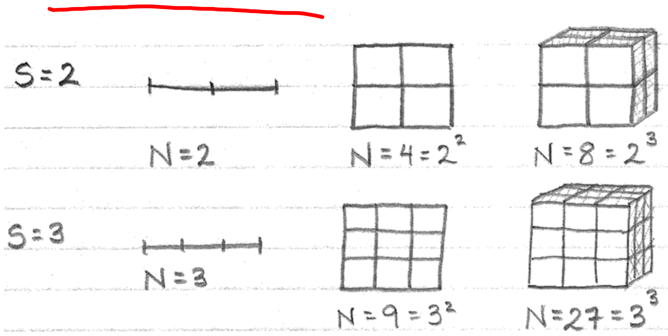


lies inside a finite area but has infinite length

Fractal Dimension:

Recall that points in space are 0-dimensional; lines are 1-dimensional; a square is 2-dimensional; and a cube is 3-dimensional. Fractals don't behave exactly like objects in these integer dimensions.

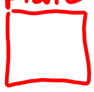
Suppose that an object has the following property: if we scale it down by a factor of **S**, then the object can be built from **N** scaled-down versions of itself.



point • 0-dim'l

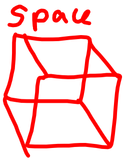
line 1-dim'l

plane



2-dim'l

space



3-dim'l

$N = S^d$, where d = dimension

$d = \frac{\log N}{\log S}$ = fractal dimension

$\log(N) = \log(S^d)$

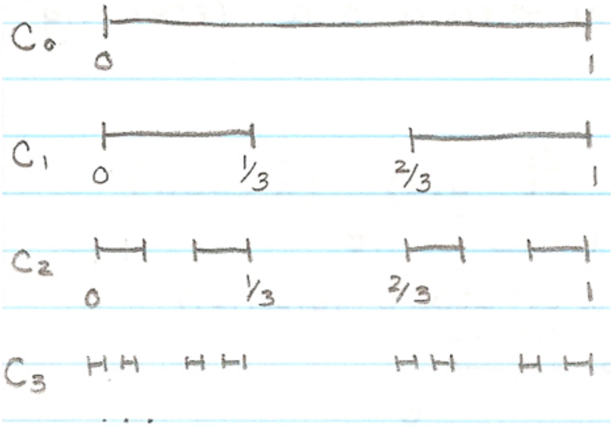
$\log N = d \cdot \log S$

$\frac{\log N}{\log S} = d$

Recall:

$\log a^p = p \log a$

The Cantor Middle-Third Set:

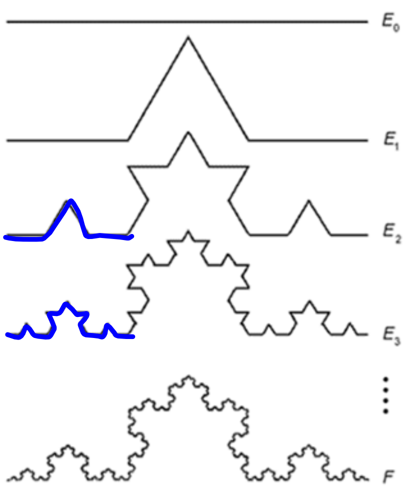


Suppose that an object has the following property: if we scale it down by a factor of **S**, then the object can be built from **N** scaled-down versions of itself.

$d = \frac{\log N}{\log S}$ = <u>**fractal dimension**</u>

<u>**Ex**</u> <u>**Cantor Set**</u> $S = 3$, $N = 2$, $d = \frac{\log 2}{\log 3} \approx \boxed{0.63}$

Koch Curve:



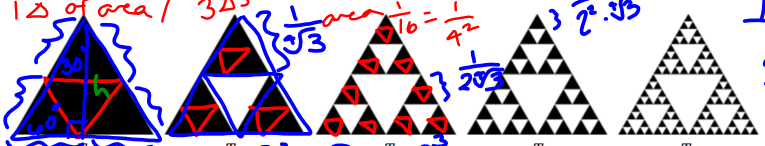
$S = 3$, $N = 4$, $d = \frac{\log 4}{\log 3} \approx \boxed{1.62}$

Sierpinski Triangle:

1 △ of area 1 / 3 △'s of area $\frac{1}{4}$ / $9 = 3^2$ △'s of area $\frac{1}{16} = \frac{1}{4^2}$

$1 = \frac{1}{2}bh$

$2 = bh$



T_0 T_1 T_2 T_3 T_4

$b = 2/\sqrt[4]{3}$ $P_1 = \frac{3^2}{2^0 \cdot \sqrt[4]{3}}$ $P_2 = \frac{3^3}{2' \cdot \sqrt[4]{3}}$ $\frac{1}{2^2 \cdot \sqrt[4]{3}}$

Let T_n be the nth iteration of the Sierpinski triangle;
let A_n be the area of the nth iteration of the Sierpinski triangle; let $A_0 = 1$;
let P_n be the perimeter of the nth iteration of the Sierpinski triangle.

area removed:

$\frac{3^0}{4^1} + \frac{3^1}{4^2} + \frac{3^2}{4^3} + \ldots = \frac{1}{4}\left(\frac{3}{4}\right)^n$

$P_0 = \frac{3^1}{2^{-1} \cdot \sqrt[4]{3}}$

T_n consists of 3^n triangles of area $\left(\frac{1}{4}\right)^n$;

How much was removed?

We remove 3^n triangles of area $\left(\frac{1}{4}\right)^{n+1}$ to get T_{n+1} from T_n.

Hence the total area removed is:

$$\sum_{n=0}^{\infty} \frac{1}{4}\left(\frac{3}{4}\right)^n = \frac{\frac{1}{4}}{1-\frac{3}{4}} = \frac{\frac{1}{4}}{\frac{1}{4}} = 1$$

$\frac{h}{\frac{1}{2}b} = \sqrt{3}$

$h = \frac{\sqrt{3}}{2}b$

$P_{n+1} = \frac{3}{2}P_n$, which implies that $P_n = \left(\frac{3}{2}\right)^n$,
which approaches ∞ as n approaches ∞.
The Sierpinski triangle has zero area but an infinite perimeter!

Fractal Dimension: $S = 2$, $N = 3$, $d = \frac{\log 3}{\log 2} \approx 1.58$

	T_0	T_1
Perimeter of this iteration	$P_0 = \frac{6}{\sqrt[4]{3}}$	$P_1 = \frac{9}{\sqrt[4]{3}}$
Perimeter of this iteration divided by perimeter of previous iteration	N/A	$\frac{P_1}{P_0} = \frac{3}{2}$
Area removed from previous iteration to obtain this iteration	N/A	$\frac{1}{4}$
Total area removed up to this point	0	$\frac{1}{4}$
Number of triangles in this iteration	1	3
Area of a triangle in this iteration	1	1/4
Total area of this iteration	$A_0 = 1$	$A_1 = 3/4$

$2 = b \cdot h = b \cdot \frac{\sqrt{3}}{2} b$

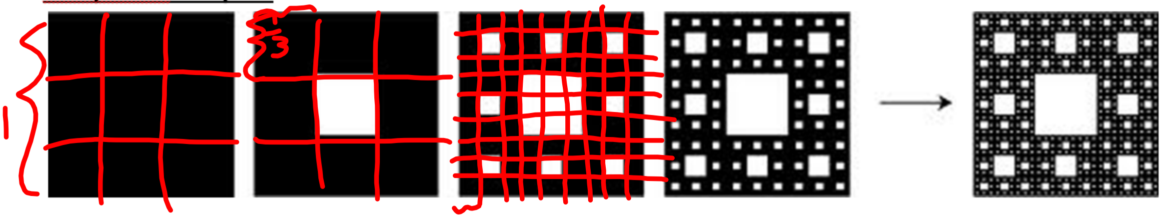
$2 = \frac{\sqrt{3}}{2} b^2$

$\frac{4}{\sqrt{3}} = b^2$

$b = \sqrt{\frac{4}{\sqrt{3}}} = \frac{\sqrt{4}}{\sqrt{\sqrt{3}}} = \frac{2}{(3^{1/2})^{1/2}} = \frac{2}{3^{1/4}}$

side length $= \frac{2}{\sqrt[4]{3}}$

Sierpinski Carpet:



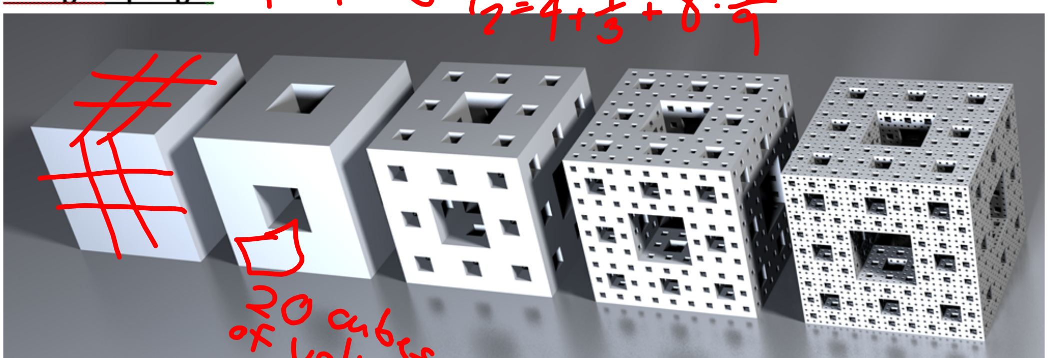
S_0 S_1 S_2 S_3

1 sq of area 1 / $P_0 = 4$

8 sq of area 1/9 / $P_1 = 4 + \frac{4}{3}$

8^2 sq of area $\frac{1}{81}$ / $P_2 = 4 + \frac{4}{3} + 8 \cdot \frac{4}{9}$

Menger Sponge:



20 cubes of volume $\left(\frac{1}{3}\right)^3$