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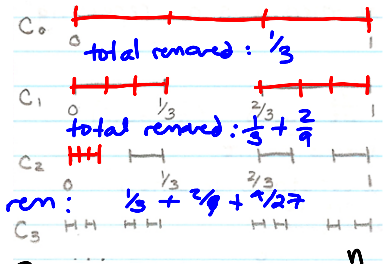
Mon. 11/10 finish fractal lecture
Tues. 11/11 fractal problem set work day
Wed. 11/12 fractal problem set due; fractal essay work day
Fri. 11/14 fractal essay due

If you haven't already, start reading articles in my Google Drive folder, search the Alabama Virtual Library, and peruse textbooks in S201 and the ASMS Library!

<u>The Cantor Middle-Third Set:</u>



2^0 1 segment of length $1 = \left(\frac{1}{3}\right)^0$

2^1 2 segments of length $\frac{1}{3} = \left(\frac{1}{3}\right)^1$

2^2 4 segments of length $\frac{1}{9} = \left(\frac{1}{3}\right)^2$

2^3 8 segments of length $\frac{1}{27} = \left(\frac{1}{3}\right)^3$

C_n consists of 2^n segments, each of length $\left(\frac{1}{3}\right)^n$

To obtain C_{n+1} from C_n, we remove the middle third of each interval in C_n. The Cantor set C is the intersection of all C_n. C is a fractal.

$2^n \cdot \left(\frac{1}{3}\right)^n = \left(\frac{2}{3}\right)^n$

C_n consists of 2^n closed intervals of length $\frac{1}{3^n}$. The total length of C_n is $\left(\frac{2}{3}\right)^n$, which approaches 0 as n approaches ∞. Hence the "length" of C is 0.

Another way to state this is that the length of $C_{n+1} = \frac{2}{3} \cdot length\ of\ C_n$. Given this recursive definition, again we have that the length of C_n is , which approaches 0 as n approaches ∞.

The total length <u>removed</u> from the interval [0 , 1] in the construction of the Cantor set is

geometric series w/ common ratio 2/3

$$\frac{1}{3}+\frac{2}{9}+\frac{4}{27}+\cdots=\sum_{n=0}^{\infty}\frac{2^n}{3^{n+1}}=\sum_{n=0}^{\infty}\frac{1}{3}\left(\frac{2}{3}\right)^n=\frac{\frac{1}{3}}{1-\frac{2}{3}}=\frac{\frac{1}{3}}{\frac{1}{3}}=\boxed{1}$$

$\frac{2^0}{3^1}+\frac{2^1}{3^2}+\frac{2^2}{3^3}$ $3^{n+1} = 3 \cdot 3^n$

Hence we have a set from which its entire length has been removed. Yet there are still infinitely many points left in the set. <u>Which points are they</u>?

the endpoints of all the intervals!

Note: the sum of an infinite geometric series with common ratio less than 1 in absolute value is equal to $\boxed{S_\infty = \frac{a_1}{1-r}}$, where a_1 is the first term and r is the common ratio.

Fractals, cont.

1 segment of length 1

Koch Curve:

$\ell_0 = 1$

4 seg. ea. $\ell = \frac{1}{3}$

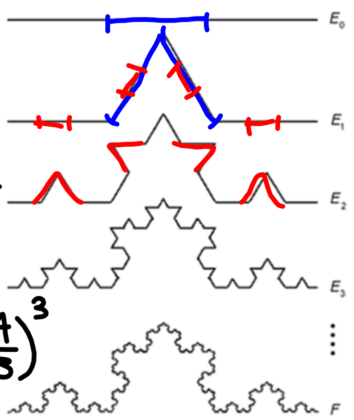
$\ell_1 = \frac{4}{3}$

16 seg. of $\ell = \frac{1}{9}$

$\ell_2 = \frac{16}{9} = \left(\frac{4}{3}\right)^2$

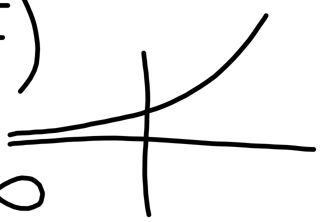
64 seg of $\ell = \frac{1}{27}$

$\ell_3 = \frac{64}{27} = \left(\frac{4}{3}\right)^3$

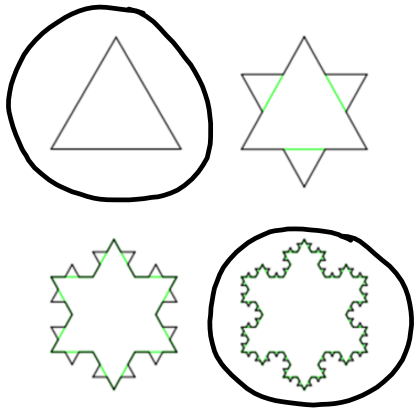


$\ell_n = \left(\frac{4}{3}\right)^n$

$\ell_K = \infty$



Koch Snowflake:

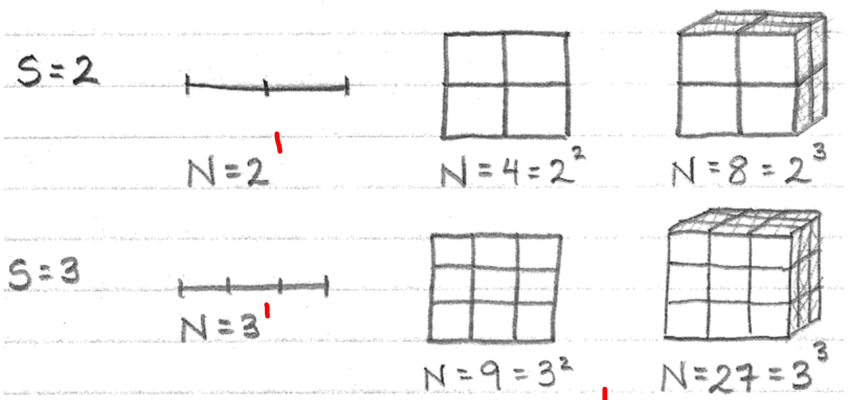


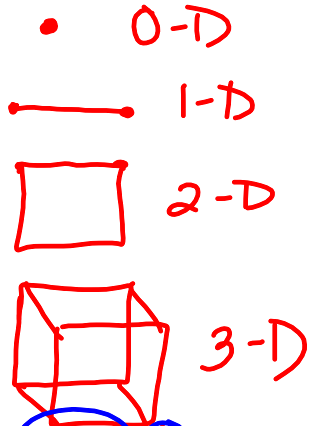
curve of infinite perimeter enclosing a finite area

Fractal Dimension:

Recall that points in space are 0-dimensional; lines are 1-dimensional; a square is 2-dimensional; and a cube is 3-dimensional. Fractals don't behave exactly like objects in these integer dimensions.

Suppose that an object has the following property: if we scale it down by a factor of **S**, then the object can be built from **N** scaled-down versions of itself.





$N = S^d$, where $d =$ dimension

$d = \frac{\log N}{\log S} =$ fractal dimension

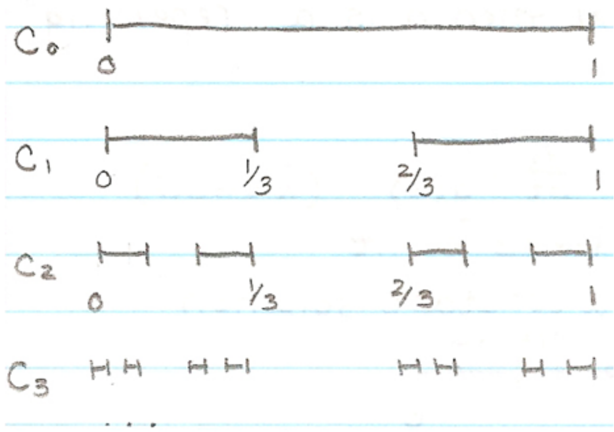
$N = S^d$

$\ln(N) = \ln(S^d)$

$\ln N = d \cdot \ln S$

$\boxed{d = \frac{\ln N}{\ln S}}$

The Cantor Middle-Third Set:



scaling factor

$S = 3$

$N = 2$

It takes 2 shrunk-down versions of C_{n-1} to make up C_n

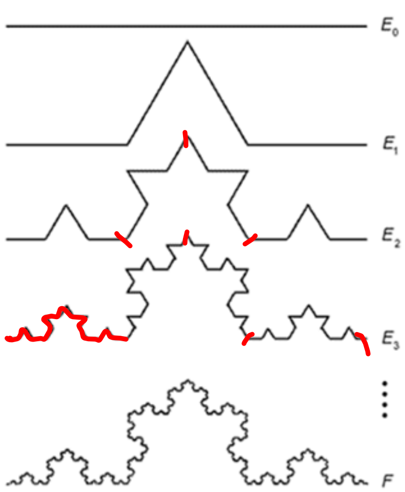
Suppose that an object has the following property: if we scale it down by a factor of **S**, then the object can be built from **N** scaled-down versions of itself.

$d = \frac{\log N}{\log S}$ = <u>fractal dimension</u>

$$d = \frac{\ln N}{\ln S} = \frac{\ln 2}{\ln 3} = 0.63$$

<u>Ex</u> <u>Cantor Set</u> $S = 3$, $N = 2$, $d = \frac{\log 2}{\log 3} \approx 0.63$

Koch Curve:



$S = 3$

$N = 4$

$$d = \frac{\ln N}{\ln S} = \frac{\ln 4}{\ln 3} \approx 1.62$$

$S = 3$, $N = 4$, $d = \frac{\log 4}{\log 3} \approx 1.62$

Sierpinski Triangle:

T_0 T_1 T_2 T_3 T_4

Let T_n be the nth iteration of the Sierpinski triangle;
let A_n be the area of the nth iteration of the Sierpinski triangle; let $A_0 = 1$;
let P_n be the perimeter of the nth iteration of the Sierpinski triangle.

T_n consists of 3^n triangles of area $\left(\frac{1}{4}\right)^n$;

How much was removed?

We remove 3^n triangles of area $\left(\frac{1}{4}\right)^{n+1}$ to get T_{n+1} from T_n.

Hence the total area removed is:

$$\sum_{n=0}^{\infty} \frac{1}{4}\left(\frac{3}{4}\right)^n = \frac{\frac{1}{4}}{1-\frac{3}{4}} = \frac{\frac{1}{4}}{\frac{1}{4}} = 1$$

$P_{n+1} = \frac{3}{2}P_n$, which implies that $P_n = \left(\frac{3}{2}\right)^n$,

which approaches ∞ as n approaches ∞.

The Sierpinski triangle has zero area but an infinite perimeter!

Fractal Dimension: $S = 2$, $N = 3$, $d = \frac{\log 3}{\log 2} \approx 1.58$

	T_0	T_1
	To	T_1
~~Perimeter of this iteration~~	~~$P_0 = \frac{6}{\sqrt[4]{3}}$~~	~~$P_1 = \frac{9}{\sqrt[4]{3}}$~~
~~Perimeter of this iteration divided by perimeter of previous iteration~~	~~N/A~~	~~$\frac{P_1}{P_0} = \frac{3}{2}$~~
Area removed from previous iteration to obtain this iteration	N/A	1/4
Total area removed up to this point	0	1/4
Number of triangles in this iteration	1	3
Area of a triangle in this iteration	1	1/4
Total area of this iteration	$A_0 = 1$	$A_1 = 3/4$

T_0 T_1 T_2 T_3 T_4 T_n

	T_0	T_1	T_2	T_3	T_4
~~Perimeter of this iteration~~	~~$P_0 = \frac{6}{\sqrt[4]{3}}$~~	~~$P_1 = \frac{9}{\sqrt[4]{3}}$~~	~~$P_2 =$~~	~~$P_3 =$~~	~~$P_4 =$~~
~~Perimeter of this iteration divided by perimeter of previous iteration~~	~~N/A~~	~~$\frac{P_1}{P_0} = \frac{3}{2}$~~	~~$\frac{P_2}{P_1} =$~~	~~$\frac{P_3}{P_2} =$~~	~~$\frac{P_4}{P_3} =$~~
Area removed from previous iteration to obtain this iteration	N/A	$\frac{1}{4} = \frac{3^0}{4^1}$	$\frac{3}{16} = \frac{3^1}{4^2}$	$\frac{9}{64} = \frac{3^2}{4^3}$	$3^3/4^4$; $\frac{3^{n-1}}{4^n}$
Total area removed up to this point	0	1/4	$\frac{3^0}{4^1} + \frac{3^1}{4^2}$	$\frac{3^0}{4^1} + \frac{3^1}{4^2} + \frac{3^2}{4^3}$	$\sum_{i=0}^{3} \frac{3^i}{4^{i+1}}$; $\sum_{i=0}^{n-1} \frac{3^i}{4^{i+1}}$
Number of triangles in this iteration	1	3	$9 = 3^2$	$27 = 3^3$	$81 = 3^4$; 3^n
Area of a triangle in this iteration	1	1/4	$\frac{1}{16} = \frac{1}{4^2}$	$\left(\frac{1}{4}\right)^3$	$\left(\frac{1}{4}\right)^4$; $\left(\frac{1}{4}\right)^n$
Total area of this iteration	$A_0 = 1$	$A_1 = 3/4$	$A_2 = \frac{9}{16} = \left(\frac{3}{4}\right)^2$	$A_3 = \left(\frac{3}{4}\right)^3$	$A_4 = \left(\frac{3}{4}\right)^4$; $\left(\frac{3}{4}\right)^n$
Area of this iteration divided by area of previous iteration	N/A	$\frac{A_1}{A_0} = 3/4$	$\frac{A_2}{A_1} = 3/4$	$\frac{A_3}{A_2} = 3/4$	$\frac{A_4}{A_3} = 3/4$

The initial triangle T_0 had area $A_0 = 1$.

Total area removed up to the n^{th} iteration is $\sum_{i=0}^{n-1} \frac{3^i}{4^{i+1}}$

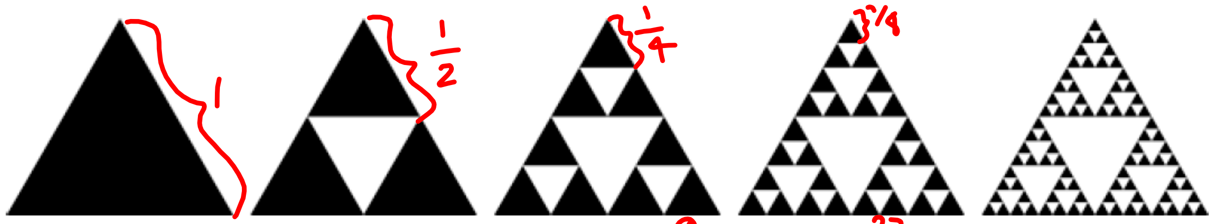
$$= \sum_{i=1}^{n} \frac{3^{i-1}}{4^i}$$

$\frac{3^{1-1}}{4^1} = \frac{3^0}{4} = \frac{1}{4}$ $\quad \frac{3^{n-1}}{4^n}$

$\frac{3^0}{4^{0+1}} = \frac{1}{4}$ $\quad \frac{3^{n-1}}{4^{n-1+1}} = \frac{3^{n-1}}{4^n}$

$$S_\infty = \frac{a_1}{1-r} = \frac{\frac{1}{4}}{1-\frac{3}{4}} = \frac{\frac{1}{4}}{\frac{1}{4}} = 1$$

We remove area 1



$+\frac{3}{2}$ $+\frac{9}{4}$ $+\frac{27}{8}$

	T_0	T_1	T_2	T_3	T_4
Perimeter of this iteration	$P_0 = 3 = \frac{3}{2^0}$	$P_1 = \frac{9}{2} = \frac{3^2}{2^1}$	$P_2 = \frac{27}{4} = \frac{3^3}{2^2}$	$P_3 = \frac{81}{8} = \frac{3^4}{2^3}$	$P_4 = \frac{3^5}{2^4}$
Perimeter of this iteration divided by perimeter of previous iteration	N/A	$\frac{P_1}{P_0} = \frac{3}{2}$	$\frac{P_2}{P_1} = \frac{3}{2}$	$\frac{P_3}{P_2} = \frac{3}{2}$	$\frac{P_4}{P_3} = \frac{3}{2}$
Area removed from previous iteration to obtain this iteration	N/A				
Total area removed up to this point	0				
Number of triangles in this iteration	1				
Area of a triangle in this iteration	1				
Total area of this iteration	$A_0 = 1$	$A_1 =$	$A_2 =$	$A_3 =$	$A_4 =$
Area of this iteration divided by area of previous iteration	N/A	$\frac{A_1}{A_0} =$	$\frac{A_2}{A_1} =$	$\frac{A_3}{A_2} =$	$\frac{A_4}{A_3} =$

$$\boxed{P_n = \frac{3^{n+1}}{2^n}}$$

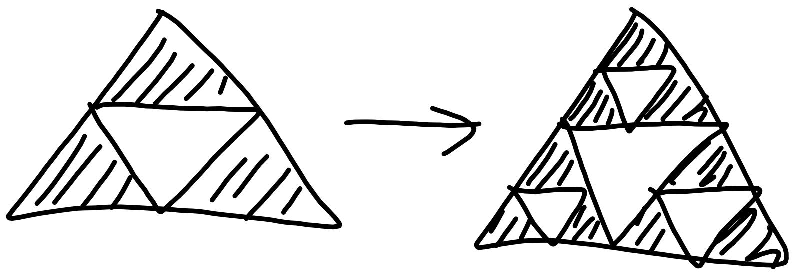
$$P_\infty = 3 + \sum_{i=1}^{\infty} \left(\frac{3}{2}\right)^n$$

$$P_\infty = 3 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \ldots + \left(\frac{3}{2}\right)^n$$

$$P_n = \frac{3^{n+1}}{2^n} \longrightarrow \ ? \quad \text{as } n \to \infty$$

$$= \frac{3 \cdot 3^n}{2^n} = 3\left(\frac{3}{2}\right)^n \longrightarrow \infty$$

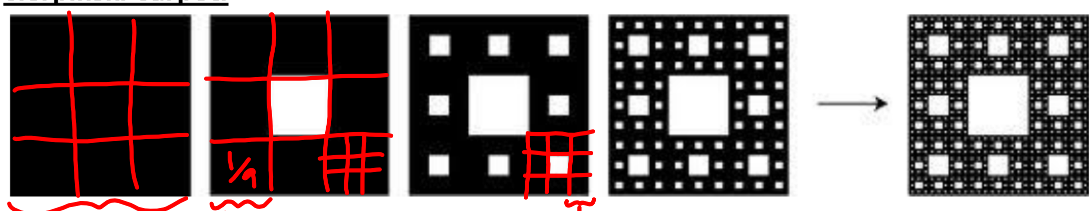
Sierpinski △ has
no area, but
infinite perimeter!!



$$s = 2 \; ; \; n = 3$$

$$d = \frac{\ln n}{\ln s} = \frac{\ln 3}{\ln 2} \approx 1.58$$

Sierpinski Carpet:



$\underbrace{\qquad}_{1}$ S_0 $\underbrace{\quad}_{1/3}$ S_1 S_2 $\underbrace{\,}_{1/9}$ S_3

Let S_n be the nth step of the construction of the Sierpinski Carpet. Suppose that S_0 has area 1.

	S_0	S_1	S_2	S_3
Perimeter of this iteration	$P_0 = 4$	$P_1 = 4 + \frac{4}{3}$	$P_2 = 4 + \frac{4}{3} + \frac{32}{9}$	$P_3 =$
Perimeter added to previous iteration to obtain this one	N/A	$4/3$	$8(4)(\frac{1}{9})$ # of squares, edges per sq., edge length	
Area removed from previous iteration to obtain this iteration	N/A	$1/9$	$\frac{8}{81}$	
Total area removed up to this point	0	$1/9$	$1/9 + 8/81$	
Number of squares in this iteration	1	8	64	
Area of a square in this iteration	1	$1/9$	$\frac{1}{81}$	
Total area of this iteration	$A_0 = 1$	$A_1 = \frac{8}{9}$	$A_2 = \frac{64}{81}$	$A_3 =$
Area of this iteration divided by area of previous iteration	N/A	$\frac{A_1}{A_0} = \frac{8}{9}$	$\frac{A_2}{A_1} = \frac{8}{9}$	$\frac{A_3}{A_2} =$