$\tau_5 \subsetneq \tau_1 \subsetneq \tau_2 \subsetneq \tau_4$

$\tau_3 \not\subseteq \tau_4$

τ_3 & τ_5 are not comparable

$\tau_3 \subsetneq \tau_1$

$\tau_1 = \{(a,b) \mid a<b\}$

$\tau_3 = \{U \mid X-U \text{ is finite or all of } X\}$

Let $U \in \tau_3$, $U \neq \emptyset$, $\mathbb{R}$

$\mathbb{R} - U$ is finite, say $\mathbb{R} - U = \{x_1, x_2, \ldots, x_n\}$ for some real #'s $x_1 < x_2 < \ldots < x_n$.

$\Rightarrow U = (-\infty, x_1) \cup (x_1, x_2) \cup (x_2, x_3) \cup \ldots \cup (x_n, \infty)$

Given $x \in U$, we can find $(a,b) \in \tau_1$ s.t.

$x \in (a,b) \subseteq U \Rightarrow \tau_3 \subseteq \tau_1$

But, given $\frac{1}{2} \in (0,1) \in \tau_1$, $\nexists U \in \tau_3$ s.t. $\frac{1}{2} \in U \subseteq (0,1)$.

To show that $\tau_a \subseteq \tau_b$,
take $U_a \in \tau_a$ & $x \in U_a$
& find $U_b \in \tau_b$ such that
$x \in U_b \subseteq U_a$

To show τ is a topology:

1. $\phi, X \in \tau$
2. $\bigcup_\alpha U_\alpha \in \tau$ $(U_\alpha \in \tau)$
3. $\bigcap_{i=1}^{n} U_i \in \tau$ $(U_i \in \tau)$

3. $\tau = \{U \mid X - U \text{ is countable or all of } X\}$

$\bigcup_\alpha U_\alpha \in \tau$?

$X - \bigcup_\alpha U_\alpha$ countable or all of X?

$A-(B\cup C)=(A-B)\cap(A-C)$ complement of unions is intersection of complements

$X-\bigcup_\alpha U_\alpha = \bigcap_\alpha (X-U_\alpha)$

$A-(B\cap C)=(A-B)\cup(A-C)$ complement of intersections is union of complements

$X-\hat{\bigcap_{i=1}} U_i = \hat{\bigcup_{i=1}}(X-U_i)$

$\tau_a = \{\{a,b,c\},\{a\},\emptyset\}$

$\tau_b = \{\{a,b,c\},\{b\},\emptyset\}$

$\tau_a \cup \tau_b = \{\emptyset,\{a,b,c\},\{a\},\{b\}\}$

$\{\tau_\alpha\}$ - family of topologies

$\emptyset, X \in \bigcup_\alpha \tau_\alpha$

$\bigcup U_\beta \stackrel{?}{\in} \bigcup_\alpha \tau_\alpha$ for $U_\beta \in \bigcup_\alpha \tau_\alpha$

$U_\beta \in \bigcup_\alpha \tau_\alpha$ doesn't necessarily mean that $U_\beta \in \tau_\alpha$ for some α

Suppose τ_0 is a topology on X containing all collections τ_α

$\Rightarrow$ either $\bigcup_\alpha \tau_\alpha \subsetneq \tau_0$

or $\bigcup_\alpha \tau_\alpha = \tau_0$