Topology Homework #2 Due: Wed. 03/18 Section 1.2 #1a, 2a,b,c,d, 4a,b,c,d,e,f

Relevant definitions and theorems:

A **function** $f: A \to B$ is a rule of assignment (which assigns to every input exactly one output) together with a set B (range) that contains the image set of the rule.

$f: A \to B$ is **injective** (or one-to-one) if given any elements $a_1, a_2 \in A, f(a_1) = f(a_2)$ implies that $a_1 = a_2$. This definition can also be stated as its contrapositive: Given any elements $a_1, a_2 \in A$ such that $a_1 \neq a_2$, if this implies that $f(a_1) \neq f(a_2)$, then f is injective.

f is **surjective** (or onto) if given any $b \in B$, there exists some $a \in A$ such that $f(a) = b$. (i.e., if given any element in the set B, you can find some element in the set A that maps to it.)

f is **bijective** if it is both injective and surjective. If f is a bijective function, then there exists a function $f^{-1}: B \to A$ called the **inverse function** of f, defined as $f^{-1}(b) = a$ if and only if $f(a) = b$.

Given a function $f: A \to B$, and $A_0 \subseteq A$, the **image** of A_0 under f is $f(A_0) = \{b \in B | b = f(a) for\ at\ least\ one\ a \in A_0\}$.

Given a function $f: A \to B$, and $B_0 \subseteq B$, the **preimage** of B_0 under f is $f^{-1}(B_0) = \{a \in A | f(a) \in B_0\}$. Note that if there are no points a of A whose images lie in B_0, then the set $f^{-1}(B_0)$ is empty.

Given functions $f: A \to B$ and $g: B \to C$, the **composition** of f and g, denoted by $g \circ f$, is defined as $(g \circ f)(a) = g(f(a))$. Formally, $g \circ f: A \to C$ is the function whose rule is $\{(a, c) | For\ some\ b \in B, f(a) = b\ and\ g(b) = c\}$.

Lemma 2.1: Let $f: A \to B$ be a function. If $\exists\ g: B \to A\ \&\ h: B \to A$ such that $g(f(a)) = a\ \ \forall a \in A$ and $f(h(b)) = b\ \ \forall b \in B$, then f is bijective and $g = h = f^{-1}$
Translation: If a function has an inverse, then that function is bijective.

Example Problem #1(b):
Let $f: A \to B$. Let $A_0 \subseteq A$ and $B_0 \subseteq B$. Show that $f(f^{-1}(B_0)) \subseteq B_0$ and that equality holds if f is surjective (onto).

Proof:

$\subseteq$ **Let $y \in f(f^{-1}(B_0))$.**
< Let y be an element in the image of the set $f^{-1}(B_0)$ under f. >
Then $y = f(a)$ for some $a \in f^{-1}(B_0)$.
< There exists an element a in the preimage of the set B_0 under f such that $y = f(a)$. >
$\Rightarrow f(a) \in B_0$.
< By definition, since a is an element of the preimage of B_0 under f, this implies that the image of a under f lies in B_0. >
Since $y = f(a)$, we have $y \in B_0$ and hence $f(f^{-1}(B_0)) \subseteq B_0$.

Assuming, in addition, that f is surjective:

$\supseteq$ **Let $x \in B_0$.**
Since $B_0 \subseteq B$ and f is surjective from A onto B, there exists at least one $a \in A$ such that $f(a) = x$.
$f(a) = x \Rightarrow f(a) \in B_0 \Rightarrow a \in f^{-1}(B_0)$
< Since the image of the element a under f is an element of the set B_0, this is precisely what it means for an element to be in the preimage of the set B_0 under f. >
By definition of the image of set $f^{-1}(B_0)$ under f, we have that $f(a) \in f(f^{-1}(B_0))$.
Since $x = f(a)$, we have $x \in f(f^{-1}(B_0))$ and hence $B_0 \subseteq f(f^{-1}(B_0))$.