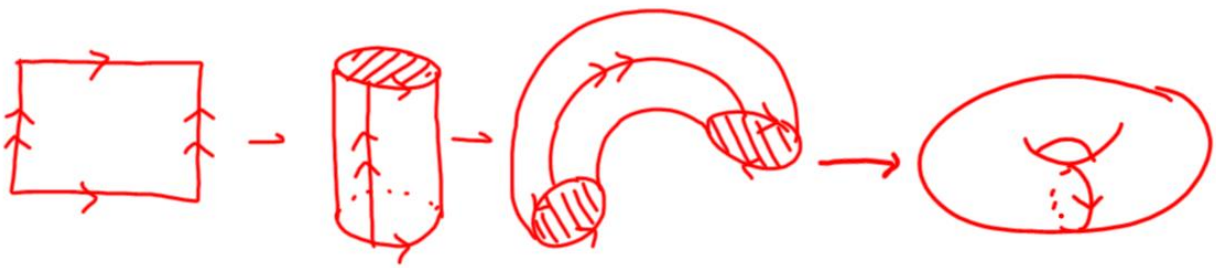
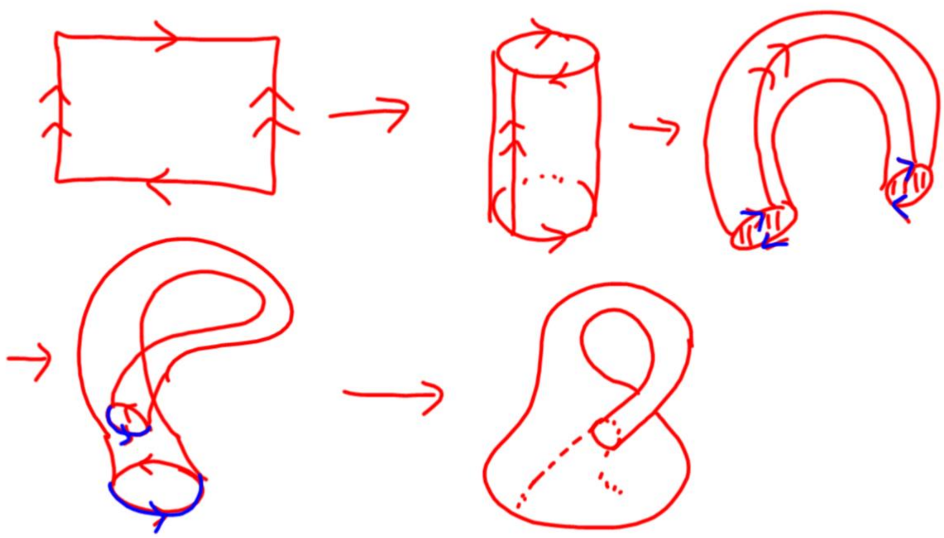
Classification of Surfaces

By identifying opposite sides of a rectangle, you can create an orientable surface of genus 1, the torus.

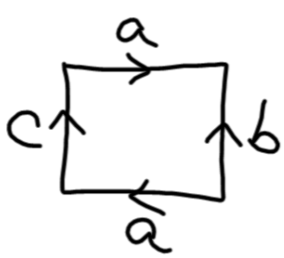


If the orientations don't match up for one of the pairs of sides, the cylinder must pass through itself, resulting in a nonorientable surface of genus 2, the Klein bottle.



All we needed to create/identify these two surfaces was a knowledge of which edges were identified with each other and with what orientation.

Side identification and orientation can be given in the form of a word, e.g. $ab^{-1}ac$. Four letters tell us to draw a square, one letter for each side. Start at one vertex, and write the word around the polygon. The -1 exponent on the b indicates opposite orientation. It does not matter whether you write the word clockwise or counter-clockwise, but orientations must be consistent.

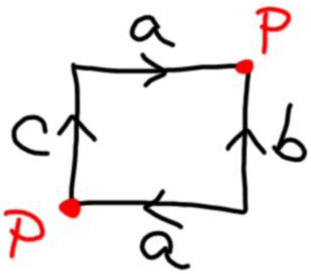


Sides with the same letter are identified.

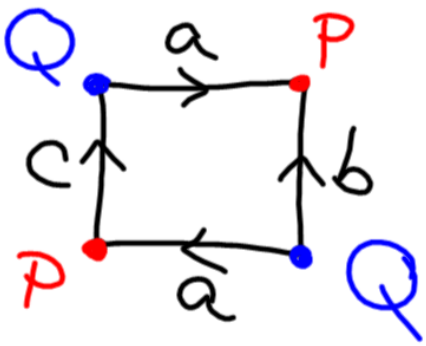
Now, we label the vertices.

Choose one vertex and call it P.

Since this point is at the head of edge c and the tail of edge a, any other vertex at the head of an edge c (or c^{-1}) or the tail of an edge a (or a^{-1}) will also be labeled P. Since the second vertex P is at the tail of edge b^{-1}, check for any other vertices at the tail of edges b or b^{-1}. Continue this process until all vertices P have been identified.



Choose an unlabeled vertex and call it Q. Repeat process used for vertex P.



Continue with R, S, etc. until all vertices have been labeled.

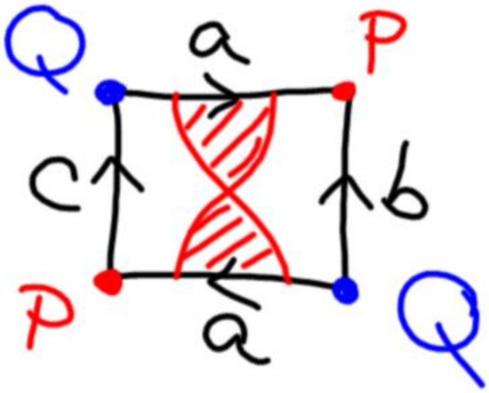
Since all "a" edges, "P" vertices, and "Q" vertices have been identified, we can now calculate the Euler characteristic, $\chi(S) = v - e + f$.

We have 1 face, 3 edges, and 2 vertices, so $\chi(S) = 2 - 3 + 1 = 0$.

Next, we will take all unidentified edges (in this case, b & c) and determine the number of boundary components. Since b^{-1} and c both connect to vertices P & Q as shown, we have 1 boundary component.

Now, we can find the genus.

Since the "a" edges have opposite orientation, we must put a twist (or Möbius band) in the surface to identify them. Any surface requiring one or more Möbius band is non-orientable. The formula for non-orientable surfaces is $\chi(S) = 2 - g - b$.

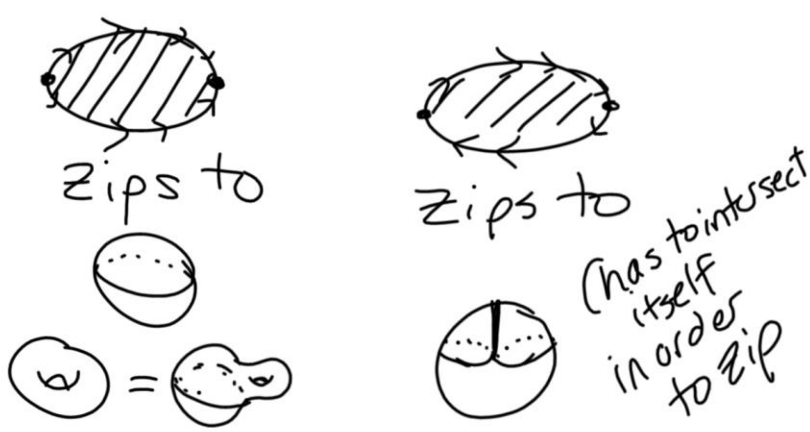


Here, $\chi(S) = 0$ and $b = 1$, so $0 = 2 - g - 1$ or $g = 1$. The word $ab^{-1}ac$ describes a non-orientable surface of genus 1 with 1 boundary component, the Möbius band.

Möbius band

The genus of a surface refers to the number of handles (for orientable surfaces) or cross-caps (for non-orientable surfaces) present in the surface.

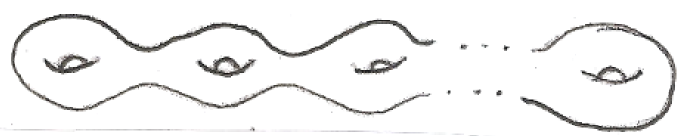
Handle v. crosscap



Sphere – orientable, no handles, genus 0

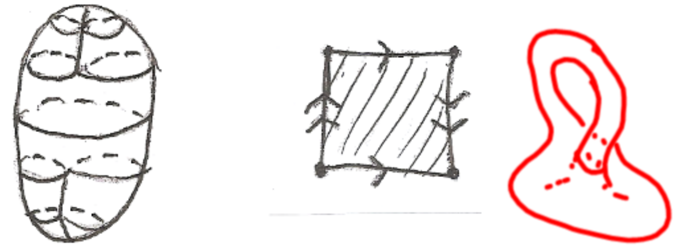
Torus – orientable, one handle, genus 1

K-torus – orientable, k handles, genus k

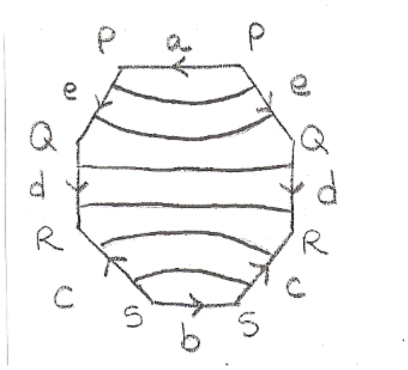


Projective plane – non-orientable, one cross-cap, genus 1

Klein bottle – non-orientable, 2 cross-caps, genus 2



Let's look at the word $aedc^{-1}bcd^{-1}e^{-1}$



1 face, 5 edges, 4 vertices →

Euler characteristic $\chi(S) = v - e + f = 4 - 5 + 1 = 0$

No Möbius bands → orientable → $\chi(S) = 2 - 2g - b \rightarrow g = 0$

This is an orientable surface of genus 0 (no handles → sphere) with 2 boundary components, the annulus.

<u>Summary of Variables and Formulas</u>

Notation:
$\chi(S)$ - Euler characteristic of the surface
b - # of boundary components (holes, perforations)
g - genus of the surface (number of handles or crosscaps)
v - # of vertices
e - # of edges
f - # of faces

Formulas:
$\chi(S) = v - e + f$ (for any surface)
$\chi(S) = 2 - 2g - b$ (for orientable surfaces)
$\chi(S) = 2 - g - b$ (for non-orientable surfaces)

<u>Assignment</u>: Identify the surface obtained by gluing the edges of a

1. hexagon according to the word $abca^{-1}b^{-1}c^{-1}$.

2. decagon according to the word $abcdec^{-1}da^{-1}b^{-1}e^{-1}$.

3. hexagon according to the word $abacb^{-1}c^{-1}$.

4. according to the word $abca^{-1}db^{-1}c^{-1}d^{-1}$.

5. decagon according to the word $ae^{-1}a^{-1}bdb^{-1}ced^{-1}c^{-1}$.