Topology Definitions and Theorems

Set Theory and Functions

(Munkres) 1.1 – Fundamental Concepts

We will use capital letters $A, B, \ldots$ to denote sets, and lowercase letters $a, b, \ldots$ to denote the objects or elements belonging to these sets.
If an object a belongs to a set A, we express this by the notation $a \in A$.
If an object a does not belong to A, we express this by the notation $a \notin A$.

We say that A is a subset of B if every element of A is also an element of B, and express this by writing $A \subseteq B$. Note that A is not required to be different from B. If $A = B$, then we have both $A \subseteq B$ and $B \subseteq A$. If $A \subseteq B$ and $A \neq B$, then we call A a proper subset of B and write $A \subset B$.

The relations $\subseteq$ and $\subset$ are called inclusion and proper inclusion. If $A \subseteq B$, we can also express this by $B \supseteq A$.

A set with a small finite number of elements can be expressed by the roster method, e.g. $A = \{a, b, c\}$

Typically, we will specify a set by one element of the set and some property that elements of the set may or may not possess, to form the set of all elements of the set having that property. e.g. $B = \{x | x\ is\ an\ even\ integer\}$ This method is often called set-builder notation.

The union of two sets A and B is the set consisting of all elements of A together with all the elements of B. Formally, $A \cup B = \{x | x \in A\ or\ x \in B\}$.

The intersection of two sets A and B is the set consisting of only the elements that A and B have in common. Formally, $A \cap B = \{x | x \in A\ and\ x \in B\}$.

The set having no elements is called the empty set, denoted by $\emptyset$.

If sets A and B have no common elements, we say that they are disjoint, and express this by $A \cap B = \emptyset$.

For every set A, we have $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$.

Contrapositive

Converse

Negation

Difference

Rules of set theory

DeMorgan's Laws

Collections of sets

Cartesian product

1.2 – Functions

Rule of assignment

Domain

Image set

Value

Image

Restriction

Lemma 2.1

Image

Preimage

Def Let $\mathcal{A}$ be a non-empty collection of sets.
An **arbitrary union** is $\bigcup_{A \in \mathcal{A}} A = \{x | x \in A \text{ for at least one } A \in \mathcal{A}\}$.
An **arbitrary intersection** is $\bigcap_{A \in \mathcal{A}} A = \{x | x \in A \text{ for every } A \in \mathcal{A}\}$

Def Let A be a set. The **power set** of A, denoted by $\mathcal{P}(A)$ is the set of all subsets of A.

Rules $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
$A - (B \cup C) = (A - B) \cap (A - C)$
$A - (B \cap C) = (A - B) \cup (A - C)$

Def A **function** $f: A \to B$ is a rule which assigns to every element of A one and only one element of B

Def Given a function $f: A \to B$, where A is the domain of f, the set $f(A) = \{b \in B | b = f(a) \text{for some } a \in A\}$ is the **range** of f.

Def The **composition** of f and g, denoted by $g \circ f$, is defined as $(g \circ f)(a) = g\big(f(a)\big)$.

Def f is **injective** (or one-to-one) if given any elements $a_1, a_2 \in A, f(a_1) = f(a_2)$ implies that $a_1 = a_2$. This definition can also be stated as its contrapositive: Given any elements $a_1, a_2 \in A$ such that $a_1 \neq a_2$, if this implies that $f(a_1) \neq f(a_2)$, then f is injective.

Def f is **surjective** (or onto) if given any $b \in B$, there exists some $a \in A$ such that $f(a) = b$. (i.e., if given any element in the set B, you can find some element in the set A that maps to it.)

Def f is **bijective** if it is both injective and surjective.

Thm The composition of bijections is a bijection.

Def If f is a bijective function, then there exists a function $f^{-1}: B \to A$ called the **inverse function** of f, defined as $f^{-1}(b) = a$ if and only if $f(a) = b$.

Lemma Let $f: A \to B$ be a function. If $\exists\ g: B \to A$ & $h: B \to A$ such that $g(f(a)) = a\ \ \forall a \in A$ and $f(h(b)) = b\ \ \forall b \in B$, then f is bijective and $g = h = f^{-1}$
Read: If a function has an inverse, then it is bijective.

Def a set A is **finite** if it is in bijection with a finite subset of $\mathbb{Z}_+$, i.e. there exists a bijection $f: A \to \{1, 2, 3, \dots, n\}$ for some $n \in \mathbb{Z}_+$.

Cor Let $B \neq \emptyset$. The following are equivalent:
1) B is finite
2) there exists a surjection $f: \{1, 2, 3, \dots, n\} \to B$
3) there exists an injection $f: B \to \{1, 2, 3, \dots, n\}$

Def A set is **infinite** if it is not finite. A set A is said to be **countably infinite** if there exists a bijection $f: A \to \mathbb{Z}_+$. A is said to be **countable** if it is finite or countably infinite. Else it is said to be **uncountable**.

Note: ∅ is finite and therefore countable

Thm Let $B \neq \emptyset$. The following are equivalent:
1) B is countable
2) there exists a surjection $f: \mathbb{Z}_+ \to B$
3) there exists an injection $f: B \to \mathbb{Z}_+$

Lemma If C is any infinite subset of $\mathbb{Z}_+$, then C is countably infinite.

Cor Every subset of a countable set is countable.

Thm Countable union of countable sets is countable

Topological Spaces

Def A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X with the following properties:
1) ∅, X are in $\mathcal{T}$
2) the union of elements in *any* subcollection of $\mathcal{T}$ is in $\mathcal{T}$

$$\bigcup_{\alpha} U_\alpha \in \mathcal{T}, \qquad U_\alpha \in \mathcal{T}\ \forall \alpha$$

3) the intersection of elements of any *finite* subcollection of $\mathcal{T}$ is in $\mathcal{T}$

$$\bigcap_{i=1}^{n} U_i \in \mathcal{T}, \qquad U_i \in \mathcal{T}\ \forall i$$

Def A set X with a specified $\mathcal{T}$ is called a **topological space**, denoted by $(X, \mathcal{T})$.

Def Let $(X, \mathcal{T})$ be a topological space, and $U \subseteq X$. U is said to be **open** if $U \in \mathcal{T}$.
*(see examples in 9-5-06 notes)

Def If X is a set, a **basis** for a topology on X is a collection $\mathfrak{B}$ of subsets of X such that:
1) for every $x \in \mathrm{X}$, $\exists$ $\mathrm{B} \in \mathfrak{B}$ such that $x \in \mathrm{B}$
2) Given $\mathrm{B}_1, \mathrm{B}_2 \in \mathfrak{B}$, if $x \in \mathrm{B}_1 \cap \mathrm{B}_2$, then $\exists$ $\mathrm{B}_3 \in \mathfrak{B}$ such that $x \in \mathrm{B}_3 \subset \mathrm{B}_1 \cap \mathrm{B}_2$

A subset U of X is said to be **open** in X (that is, to be an element of $\mathcal{T}$) if for each $x \in U$, there is a basis element $\mathrm{B} \in \mathfrak{B}$ such that $x \in \mathrm{B} \subset U$.

$\mathfrak{B}$ is a basis for $\mathcal{T}$ if $\mathcal{T}$ is **generated** by $\mathfrak{B}$, i.e. for any $U \subseteq X$, U open, and $x \in U$, $\exists$ $\mathrm{B} \in \mathfrak{B}$ such that $x \in \mathrm{B} \subset U$.

Def Let $\mathfrak{B}$ be all open intervals in $\mathbb{R}$, i.e. $(a, b) = \{x \in \mathbb{R} | a < x < b\}$. The topology generated by $\mathfrak{B}$ is the **standard topology on** $\mathbb{R}$.

Let X be a topological space and $\mathrm{Y} \subset X$. Then the collection $\mathcal{T}_Y = \{U \cap \mathrm{X} | U \textit{ open in } X\}$ is a topology on Y, called the **subspace topology**.

Closed Sets and Limit Points

Def Let X be a topological space. $\mathrm{A} \subset X$ is a **closed set** if $X - A$ ia open.
*(see examples in 9-19-06 notes)

Thm Let X be a topological space. The following hold:
1) $\emptyset$, X are closed
2) arbitrary intersections of closed sets are closed, i.e. if A_i are closed, $\bigcap_i A_i$ is closed.
3) finite unions of closed sets are closed

Def Let X be a topological space and let $\mathrm{A} \subset X$. The **closure** of A, denoted by $\bar{A}$, is the intersection of all closed sets containing A.
*(see observations in 9-19-06 notes)

Thm Let $\mathrm{A} \subset X$and let $\mathfrak{B}$ be a basis for X. Then $x \in \bar{A}$ if and only if every open set U containing x intersects A.

Def U is a **neighborhood** of x if U is open and $x \in U$

Def Let X be a topological space and let $\mathrm{A} \subset X$. $x \in X$ is said to be a **limit point** of A if every neighborhood of x intersects A in a point other than x.
(see diagram and examples in 9-21-06 notes)

Thm Let $\mathrm{A} \subset X$ and let A' be the set of all limit points of A. Then $\bar{A} = A \cup A'$.

Cor A is closed if and only if A contains all its limit points.

Def A topological space is **Hausdorff** if for each pair of distinct points x and y, there exist disjoint neighborhoods of x and y.

Thm Every finite set in a Hausdorff space is closed.

Thm Let X be a Hausdorff space and let $A \subset X$. Then x is a limit point of A if and only if every neighborhood of x contains infinitely many points of A.

Continuous Functions

Def Let X and Y be topological spaces and $f: X \to Y$ be a function. f is said to be **continuous** if for every open set V in Y, $f^{-1}(V)$ is open in X.
$f^{-1}(V) = \{x \in X | f(x) \in V\}$

Thm Let X and Y be topological spaces and $f: X \to Y$ be a function. The following are equivalent:
1) f is continuous
2) $f^{-1}(B)$ is closed for every closed set $B \subset Y$
3) for each $x \in X$ and each neighborhood V of $f(x)$, there exists a neighborhood U of x such that $f(U) \subset V$.

Def Let $f: X \to Y$ be a bijection. If both f and f^{-1} are continuous, then f is called a **homeomorphism**.
*(see examples in 9-26-06 notes)

Connectedness and Compactness

Def $A \cup B$ is a **separation** of X if
1) $A \cup B = X$
2) $A \cap B = \emptyset$
3) $A, B \neq \emptyset$
4) A, B are either both open or both closed

Def X is **disconnected** if there exists a separation
X is **connected** if it is not disconnected

Def A collection A of subsets of a topological space X is called an **open cover** of X if
1) the union of all elements of A equals X and
2) elements of A are open subsets of X.

A space X is said to be **compact** if every open cover of X has a finite subcover.

Thm Every closed subspace of a compact space is compact.

Thm Any compact subspace of a Hausdorff space is closed.

Thm Let $f: X \to Y$ be a bijective continuous function. Let X be compact and Y be Hausdorff. Then f is a homeomorphism.

Thm A continuous image of a compact space is compact.

Thm (**Heine-Borel**) A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Classification of Surfaces

Def A **surface** is a Hausdorff space with a countable basis such that every point has a neighborhood homeomorphic to the open disk in $\mathbb{R}^2$.

Def A **surface with boundary** is a Hausdorff space with countable basis such that every point has a neighborhood homeomorphic to either the open disk in $\mathbb{R}^2$ or the half disk in $\mathbb{R}^2$.

Def A **closed surface** is a compact surface with no boundary.

*(see examples in 1-9-06 notes)

Thm (**Classification theorem**) Any connected closed surface is homeomorphic to a sphere with handles or a sphere with crosscaps.

Def If S is a closed surface, then the **genus** of S is the number of handles or the number of crosscaps in S, denoted by $g(S)$. If S is not closed, cap off the boundary to get a closed surface $\bar{S}$, and $g(S) = g(\bar{S})$.

Def A surface is **orientable** if it is homeomorphic to a sphere with handles. A surface is **nonorientable** otherwise.

Def A **cell decomposition** of a surface S is a decomposition of S into a union of disks, arcs, and points. The disks are called **faces**, the arcs are **edges**, and points are **vertices**.

Def Let S be a compact surface and v, e, & f be the number of vertices, edges, and faces of a cell decomposition of S. Then the **Euler characteristic** of S, denoted by $\chi(S)$ is $\chi(S) = v - e + f$

Thm The Euler characteristic of an orientable surface S is $\chi(S) = 2 - 2g - b$, where g is the genus and b is the number of boundary components. The Euler characteristic of a non-orientable surface S is $\chi(S) = 2 - g - b$.