Set Theory and Functions

(Munkres) 1.1 – Fundamental Concepts

We will use capital letters $A, B, \ldots$ to denote sets, and lowercase letters $a, b, \ldots$ to denote the objects or elements belonging to these sets.

If an object $a$ belongs to a set $A$, we express this by the notation $a \in A$.

If an object $a$ does not belong to $A$, we express this by the notation $a \notin A$.

We say that $A$ is a subset of $B$ if every element of $A$ is also an element of $B$, and express this by writing $A \subseteq B$. Note that $A$ is not required to be different from $B$. If $A = B$, then we have both $A \subseteq B$ and $B \subseteq A$. If $A \subseteq B$ and $A \neq B$, then we call $A$ a proper subset of $B$ and write $A \subset B$.

The relations $\subseteq$ and $\subset$ are called inclusion and proper inclusion. If $A \subseteq B$, we can also express this by $B \supseteq A$.

A set with a small finite number of elements can be expressed by the roster method, e.g. $A = \{a, b, c\}$

Typically, we will specify a set by one element of the set and some property that elements of the set may or may not possess, to form the set of all elements of the set having that property. e.g. $B = \{x | x$ is an even integer$\}$ This method is often called set-builder notation.

The union of two sets $A$ and $B$ is the set consisting of all elements of $A$ together with all the elements of $B$. Formally, $A \cup B = \{x | x \in A \ or \ x \in B\}$.

The intersection of two sets $A$ and $B$ is the set consisting of only the elements that $A$ and $B$ have in common. Formally, $A \cap B = \{x | x \in A \ and \ x \in B\}$.

The set having no elements is called the empty set, denoted by $\emptyset$.

If sets $A$ and $B$ have no common elements, we say that they are disjoint, and express this by $A \cap B = \emptyset$.

For every set $A$, we have $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$.

Contrapositive

Converse

Negation

Difference

Rules of set theory

DeMorgan’s Laws

Collections of sets

Cartesian product
1.2 – Functions

Rule of assignment

Domain

Image set

Value

Image

Restriction

Lemma 2.1

Image

Preimage

**Def** Let \( \mathcal{A} \) be a non-empty collection of sets.

An **arbitrary union** is \( \bigcup_{A \in \mathcal{A}} A = \{x \mid x \in A \text{ for at least one } A \in \mathcal{A}\} \).

An **arbitrary intersection** is \( \bigcap_{A \in \mathcal{A}} A = \{x \mid x \in A \text{ for every } A \in \mathcal{A}\} \)

**Def** Let \( A \) be a set. The **power set** of \( A \), denoted by \( \mathcal{P}(A) \) is the set of all subsets of \( A \).

**Rules**

\[
\begin{align*}
\bigcup (B \cap C) &= (A \cup B) \cap (A \cup C) \\
A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) \\
A - (B \cup C) &= (A - B) \cap (A - C) \\
A - (B \cap C) &= (A - B) \cup (A - C)
\end{align*}
\]

**Def** A **function** \( f : A \to B \) is a rule which assigns to every element of \( A \) one and only one element of \( B \)

**Def** Given a function \( f : A \to B \), where \( A \) is the domain of \( f \), the set \( f(A) = \{b \in B \mid b = f(a) \text{ for some } a \in A\} \) is the **range** of \( f \).

**Def** The **composition** of \( f \) and \( g \), denoted by \( g \circ f \), is defined as \( (g \circ f)(a) = g(f(a)) \).

**Def** \( f \) is **injective** (or one-to-one) if given any elements \( a_1, a_2 \in A \), \( f(a_1) = f(a_2) \) implies that \( a_1 = a_2 \). This definition can also be stated as its contrapositive: Given any elements \( a_1, a_2 \in A \) such that \( a_1 \neq a_2 \), if this implies that \( f(a_1) \neq f(a_2) \), then \( f \) is injective.

**Def** \( f \) is **surjective** (or onto) if given any \( b \in B \), there exists some \( a \in A \) such that \( f(a) = b \). (i.e., if given any element in the set \( B \), you can find some element in the set \( A \) that maps to it.)

**Def** \( f \) is **bijective** if it is both injective and surjective.

**Thm** The composition of bijections is a bijection.
If $f$ is a bijective function, then there exists a function $f^{-1}: B \rightarrow A$ called the inverse function of $f$, defined as $f^{-1}(b) = a$ if and only if $f(a) = b$.

**Lemma** Let $f: A \rightarrow B$ be a function. If $\exists g: B \rightarrow A \& h: B \rightarrow A$ such that $g(f(a)) = a \ \forall a \in A$ and $f(h(b)) = b \ \forall b \in B$, then $f$ is bijective and $g = h = f^{-1}$.

**Def** A set $A$ is finite if it is in bijection with a finite subset of $\mathbb{Z}_+$, i.e. there exists a bijection $f: A \rightarrow \{1, 2, 3, \ldots, n\}$ for some $n \in \mathbb{Z}_+$.

**Cor** Let $B \neq \emptyset$. The following are equivalent:
1) $B$ is finite
2) there exists a surjection $f: \{1, 2, 3, \ldots, n\} \rightarrow B$
3) there exists an injection $f: B \rightarrow \{1, 2, 3, \ldots, n\}$

**Def** A set is infinite if it is not finite. A set $A$ is said to be countably infinite if there exists a bijection $f: A \rightarrow \mathbb{Z}_+$. $A$ is said to be countable if it is finite or countably infinite. Else it is said to be uncountable.

Note: $\emptyset$ is finite and therefore countable.

**Thm** Let $B \neq \emptyset$. The following are equivalent:
1) $B$ is countable
2) there exists a surjection $f: \mathbb{Z}_+ \rightarrow B$
3) there exists an injection $f: B \rightarrow \mathbb{Z}_+$

**Lemma** If $C$ is any infinite subset of $\mathbb{Z}_+$, then $C$ is countably infinite.

**Cor** Every subset of a countable set is countable.

**Thm** Countable union of countable sets is countable.

### Topological Spaces

**Def** A topology on a set $X$ is a collection $\mathcal{T}$ of subsets of $X$ with the following properties:
1) $\emptyset, X$ are in $\mathcal{T}$
2) the union of elements in any subcollection of $\mathcal{T}$ is in $\mathcal{T}$
3) the intersection of elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$

**Def** A set $X$ with a specified $\mathcal{T}$ is called a topological space, denoted by $(X, \mathcal{T})$.

**Def** Let $(X, \mathcal{T})$ be a topological space, and $U \subseteq X$. $U$ is said to be open if $U \in \mathcal{T}$. *(see examples in 9-5-06 notes)*
Def If $X$ is a set, a **basis** for a topology on $X$ is a collection $\mathcal{B}$ of subsets of $X$ such that:
1) for every $x \in X$, $\exists B \in \mathcal{B}$ such that $x \in B$
2) Given $B_1, B_2 \in \mathcal{B}$, if $x \in B_1 \cap B_2$, then $\exists B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$

A subset $U$ of $X$ is said to be **open** in $X$ (that is, to be an element of $\mathcal{T}$) if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B \subset U$.

$\mathcal{B}$ is a basis for $\mathcal{T}$ if $\mathcal{T}$ is **generated** by $\mathcal{B}$, i.e. for any $U \subseteq X$, $U$ open, and $x \in U$, $\exists B \in \mathcal{B}$ such that $x \in B \subset U$.

Def Let $\mathcal{B}$ be all open intervals in $\mathbb{R}$, i.e. $(a, b) = \{x \in \mathbb{R} | a < x < b\}$. The topology generated by $\mathcal{B}$ is the **standard topology on $\mathbb{R}$**.

Let $X$ be a topological space and $Y \subseteq X$. Then the collection $\mathcal{T}_Y = \{U \cap X | U \text{ open in } X\}$ is a topology on $Y$, called the **subspace topology**.

### Closed Sets and Limit Points

**Def** Let $X$ be a topological space. $A \subseteq X$ is a **closed set** if $X - A$ is open.
*(see examples in 9-19-06 notes)*

**Thm** Let $X$ be a topological space. The following hold:
1) $\emptyset, X$ are closed
2) arbitrary intersections of closed sets are closed, i.e. if $A_i$ are closed, $\bigcap_i A_i$ is closed.
3) finite unions of closed sets are closed

**Def** Let $X$ be a topological space and let $A \subseteq X$. The **closure** of $A$, denoted by $\bar{A}$, is the intersection of all closed sets containing $A$.
*(see observations in 9-19-06 notes)*

**Thm** Let $A \subseteq X$ and let $\mathcal{B}$ be a basis for $X$. Then $x \in \bar{A}$ if and only if every open set $U$ containing $x$ intersects $A$.

**Def** $U$ is a **neighborhood** of $x$ if $U$ is open and $x \in U$

**Def** Let $X$ be a topological space and let $A \subseteq X$. $x \in X$ is said to be a **limit point** of $A$ if every neighborhood of $x$ intersects $A$ in a point other than $x$.
(see diagram and examples in 9-21-06 notes)

**Thm** Let $A \subseteq X$ and let $A'$ be the set of all limit points of $A$. Then $\bar{A} = A \cup A'$.

**Cor** $A$ is closed if and only if $A$ contains all its limit points.

**Def** A topological space is **Hausdorff** if for each pair of distinct points $x$ and $y$, there exist disjoint neighborhoods of $x$ and $y$.

**Thm** Every finite set in a Hausdorff space is closed.

**Thm** Let $X$ be a Hausdorff space and let $A \subseteq X$. Then $x$ is a limit point of $A$ if and only if every neighborhood of $x$ contains infinitely many points of $A$.

### Continuous Functions
Let $X$ and $Y$ be topological spaces and $f: X \to Y$ be a function. $f$ is said to be **continuous** if for every open set $V$ in $Y$, $f^{-1}(V)$ is open in $X$.

$$f^{-1}(V) = \{ x \in X | f(x) \in V \}$$

**Thm** Let $X$ and $Y$ be topological spaces and $f: X \to Y$ be a function. The following are equivalent:

1) $f$ is continuous
2) $f^{-1}(B)$ is closed for every closed set $B \subseteq Y$
3) for each $x \in X$ and each neighborhood $V$ of $f(x)$, there exists a neighborhood $U$ of $x$ such that $f(U) \subseteq V$.

**Def** Let $f: X \to Y$ be a bijection. If both $f$ and $f^{-1}$ are continuous, then $f$ is called a **homeomorphism**. 

*(see examples in 9-26-06 notes)*

**Connectedness and Compactness**

**Def** $A \cup B$ is a **separation** of $X$ if

1) $A \cup B = X$
2) $A \cap B = \emptyset$
3) $A, B \neq \emptyset$
4) $A, B$ are either both open or both closed

**Def** $X$ is **disconnected** if there exists a separation

$X$ is **connected** if it is not disconnected

**Def** A collection $A$ of subsets of a topological space $X$ is called an **open cover** of $X$ if

1) the union of all elements of $A$ equals $X$ and
2) elements of $A$ are open subsets of $X$.

A space $X$ is said to be **compact** if every open cover of $X$ has a finite subcover.

**Thm** Every closed subspace of a compact space is compact.

**Thm** Any compact subspace of a Hausdorff space is closed.

**Thm** Let $f: X \to Y$ be a bijective continuous function. Let $X$ be compact and $Y$ be Hausdorff. Then $f$ is a homeomorphism.

**Thm** A continuous image of a compact space is compact.

**Thm** (Heine-Borel) A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

**Classification of Surfaces**

**Def** A **surface** is a Hausdorff space with a countable basis such that every point has a neighborhood homeomorphic to the open disk in $\mathbb{R}^2$.

**Def** A **surface with boundary** is a Hausdorff space with countable basis such that every point has a neighborhood homeomorphic to either the open disk in $\mathbb{R}^2$ or the half disk in $\mathbb{R}^2$.

**Def** A **closed surface** is a compact surface with no boundary.
Thm (Classification theorem) Any connected closed surface is homeomorphic to a sphere with handles or a sphere with crosscaps.

Def If $S$ is a closed surface, then the genus of $S$ is the number of handles or the number of crosscaps in $S$, denoted by $g(S)$. If $S$ is not closed, cap off the boundary to get a closed surface $\overline{S}$, and $g(S) = g(\overline{S})$.

Def A surface is orientable if it is homeomorphic to a sphere with handles. A surface is nonorientable otherwise.

Def A cell decomposition of a surface $S$ is a decomposition of $S$ into a union of disks, arcs, and points. The disks are called faces, the arcs are edges, and points are vertices.

Def Let $S$ be a compact surface and $v$, $e$, & $f$ be the number of vertices, edges, and faces of a cell decomposition of $S$. Then the Euler characteristic of $S$, denoted by $\chi(S)$ is $\chi(S) = v - e + f$

Thm The Euler characteristic of an orientable surface $S$ is $\chi(S) = 2 - 2g - b$, where $g$ is the genus and $b$ is the number of boundary components. The Euler characteristic of a non-orientable surface $S$ is $\chi(S) = 2 - g - b$. 

*(see examples in 1-9-06 notes)*