Trigonometry Notes 29 April 2011

Solving trigonometric equations where there is a coefficient in front of x

Example 1. Solve for x. (no restrictions on x → infinitely many solutions)

$\mathbf{\tan 3x = -1}$

$3x$ is the angle whose tangent value is -1. There are infinitely many such angles, so we start with those the first time around the unit circle. A tangent value of 1 or -1 implies adjacent and opposite side lengths are the same, so we are looking at a $45°$ or $\frac{\pi}{4}$ reference angle. Tangent is positive in quadrants I & III and negative in quadrants II & IV, so we want the angles in quadrants II and IV with $\frac{\pi}{4}$ reference angles. This yields:

$3x = \frac{3\pi}{4}$ and $3x = \frac{7\pi}{4}$

Remember, though, that we are looking for ALL such angles, and they will all be coterminal with these two, so we add $2\pi k$, where k is an integer, to each solution to get

$3x = \frac{3\pi}{4} + 2\pi k$ and $3x = \frac{7\pi}{4} + 2\pi k$

We want to solve for x rather than $3x$, so we now have to divide both sides of each equation by 3 to get

$$\boxed{x = \frac{\pi}{4} + \frac{2\pi k}{3} \ and \ x = \frac{7\pi}{12} + \frac{2\pi k}{3}}$$

This is a perfectly acceptable solution. Notice however, that the angles $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$ differ exactly by π, so rather than writing $3x = \frac{3\pi}{4} + 2\pi k$ and $3x = \frac{7\pi}{4} + 2\pi k$, we could simply write the single expression $3x = \frac{3\pi}{4} + \pi k$, which would cover all of the same angles as the two former expressions. This yields a much nicer, simpler solution: $\boxed{x = \frac{\pi}{4} + \frac{\pi k}{3}}$.

Example 2. Solve for x. (no restrictions on x → infinitely many solutions)

$$\mathbf{\cos\left(2x - \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}}$$

$2x - \frac{\pi}{6}$ is the angle whose cosine value is $\frac{\sqrt{3}}{2}$, and again, there are infinitely many such angles. The first time around the unit circle, they are the $\frac{\pi}{6}$ reference angles in quadrants I & IV, where cosine is positive.

This gives us $2x - \frac{\pi}{6} = \frac{\pi}{6} + 2\pi k$ and $2x - \frac{\pi}{6} = \frac{11\pi}{6} + 2\pi k$. Solving for x by adding $\frac{\pi}{6}$ to both sides and then dividing by 2, we get

$2x = \frac{\pi}{6} + \frac{\pi}{6} + 2\pi k$ and $2x = \frac{11\pi}{6} + \frac{\pi}{6} + 2\pi k$

$2x = \frac{2\pi}{6} + 2\pi k$ and $2x = \frac{12\pi}{6} + 2\pi k$

$2x = \frac{\pi}{3} + 2\pi k$ and $2x = 2\pi + 2\pi k$

$$\boxed{x = \frac{\pi}{6} + \pi k \;\; and \; x = \pi + \pi k}$$

The last equation can be written more simply as $x = \pi k$, since $\pi = \pi k$ when $k = 1$.

$$\boxed{x = \frac{\pi}{6} + \pi k \;\; and \; x = \pi k}$$

Example 3. Solve for x. (no restrictions on x → infinitely many solutions)

$\mathbf{\cos 5x = -\frac{1}{2}}$ Solve on your own.

Example 4. Solve for x. (no restrictions on x → infinitely many solutions)

$\mathbf{\sin\left(3x + \frac{\pi}{4}\right) = 1}$ Solve on your own.

Example 5. Solve for $x \in [0,2\pi)$. (note the restriction on x)

$$\mathbf{\sin 4x = \frac{1}{2}}$$

We start by finding the angles the first time around the unit circle with a sine value of $\frac{1}{2}$. An adjacent side of 1 and hypotenuse of 2 tells us these have $\frac{\pi}{6}$ reference angles, and sine is positive in quadrants I & II, so we set

$$4x = \frac{\pi}{6}\ ,\ \frac{5\pi}{6}$$

Since we're looking for $0 \leq x < 2\pi$, we need $0 \leq 4x < 8\pi$, i.e. we're looking for solutions 4 times around the unit circle so that when we divide by 4 to solve for x, we end up with all of the possible solutions between 0 and 2π. We accomplish this by adding 2π to each solution for the 2nd time around the unit circle, again for the third, and again for the fourth. Adding 2π to $\frac{\pi}{6}$ or $\frac{5\pi}{6}$ requires getting a common denominator, so we rewrite $2\pi \cdot \frac{6}{6} = \frac{12\pi}{6}$, so we're adding $\frac{12\pi}{6}$ to each solution three times (as opposed to adding 2π infinitely many, or k, times as we did in the previous examples). This yields

$$4x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{25\pi}{6}, \frac{29\pi}{6}, \frac{37\pi}{6}, \frac{41\pi}{6}$$

Finally, dividing each solution by 4 is the same as multiplying by $\frac{1}{4}$ (or multiplying the numerators by 1 and the denominators by 4), so our final solution set is

$$\boxed{x = \frac{\pi}{24}, \frac{5\pi}{24}, \frac{13\pi}{24}, \frac{17\pi}{24}, \frac{25\pi}{24}, \frac{29\pi}{24}, \frac{37\pi}{24}, \frac{41\pi}{24}}$$

Example 6. Solve for $x \in [0,2\pi)$. (note the restriction on x)

$$\mathbf{\tan 2x = 0}$$

Tangent can always be thought of as sine over cosine, and a fraction is equal to zero wherever its denominator is equal to zero, so $\tan x = 0$ the same places $\sin x = 0$. The first time around the unit circle these are at 0 and π. Then we add 2π to each of these solution once for a total of two trips around the unit circle (the coefficient in front of x always tells you how many times to go around), and then divide by 2.

$$2x = 0, \pi, 2\pi, 3\pi$$

$$\boxed{x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}}$$

Example 7. Solve for $x \in [0,2\pi)$. (note the restriction on x)

$\mathbf{\sin 3x = 0}$ Solve on your own.

Example 8. Solve for $x \in [0,2\pi)$. (note the restriction on x)

$\mathbf{\cos 4x = -1}$ Solve on your own.